How to Integrate Planck’s Function

Kevin Krisciunas

ABSTRACT

In gory detail we show how to integrate Planck’s function $B_\nu$. The hard part involves showing how a particular “undoable” integral is related to Euler’s Gamma Function and the Riemann Zeta Function. The corresponding infinite sum can be calculated exactly by working out the Fourier series for $f(x) = x^2$ and applying Parseval’s equation. This gives the Stefan-Boltzmann Law, $E = \sigma T^4$, and allows us to determine the numerical value of the Stefan-Boltzmann constant $\sigma$. The amount of radiation emitted by a black body, and it being proportional to the fourth power of the temperature constitute a fundamental building block of astrophysics.

1. Introduction

A solid, liquid, or dense gas gives off a continuous spectrum (also known as a black body spectrum). The peak of the spectral energy distribution depends on the temperature of the black body. A rock at room temperature gives off infrared radiation that peaks at a wavelength of about 10 microns. If we heated a solid with a very high melting point to 6000 K, it would be a black body source that produces mostly visible light, but it also produces significant amounts of ultraviolet and infrared light. The Sun’s photosphere has just such a temperature, but because the photosphere is cooler than the deeper layers that produce continuous radiation, light is absorbed and scattered by the atoms in the photosphere. Which photons are absorbed? It depends on the composition and ionization state of the atoms. An absorption spectrum results.

The spectral radiance of a black body at absolute temperature $T$ is given by Planck’s function (Carroll & Ostlie 2007, p. 73)\(^2\):

$$B_\nu = \frac{2h}{c^2} \frac{\nu^3}{e^{\frac{h\nu}{kT}} - 1}, \quad (1)$$

1Texas A. & M. University, Department of Physics and Astronomy, 4242 TAMU, College Station, TX 77843-4242; krisciunas@physics.tamu.edu

2See also the Wikipedia article on “Planck’s law”.

\[\]
where $\nu$ is the frequency of light, $h$ is Planck’s constant, $c$ is the velocity of light, and $k$ is Boltzmann’s constant. The units of $B_\nu$ are ergs/s/cm$^2$/sr/Hz.

A similar form of the spectral radiance, but as a function of wavelength $\lambda$, is:

$$B_\lambda = \frac{2hc^2/\lambda^5}{e^{hc/\lambda kT} - 1}.$$  \hspace{1cm} (2)

Here the units are ergs/s/cm$^3$/sr.

We can multiply Eqs. 1 and 2 by $4\pi/c$ to give the spectral energy density $u_\nu$ or $u_\lambda$, which is measured in terms of energy per unit volume per spectral unit.

In Fig. 1 we see an example of the spectral energy density of a black body of temperature 6000 K, both in frequency form and wavelength form. While each function clearly has a maximum, and for an individual photon $\nu = c/\lambda$, it is not the case that $\nu_{\text{max}} = c/\lambda_{\text{max}}$.

From radiative transfer theory we note that the amount of radiation emitted by a black body at a particular frequency will be

$$F_\nu = \int_0^{2\pi} d\phi \int_{-1}^1 I \mu d\mu,$$  \hspace{1cm} (3)

where $\mu = \cos \theta$. $I = B_\nu$ for $\mu$ ranging from 0 to +1. $I = 0$ for $\mu$ ranging from $-1$ to 0. This means that we only need to consider photons emitted in the $2\pi$ steradians between the source and the observer. Photons emitted in the other direction are not received by the observer. We then have

$$F_\nu = 2\pi B_\nu \left( \frac{\mu^2}{2} \right)_{-1}^1 = \pi B_\nu.$$  \hspace{1cm} (4)

Integrating $B_\nu$ over all frequencies and multiplying by $\pi$ gives the total amount of energy given off by the black body each second. This is known as the Stefan-Boltzmann Law:

$$E = \pi \int_0^\infty B_\nu d\nu = \sigma T^4,$$  \hspace{1cm} (5)

where $\sigma$ is the Stefan-Boltzmann constant. In this writer’s opinion, if you want to call yourself an astrophysicist, at least once you should go through the full derivation of integrating Eq. 5, which is the purpose of this note. This is non-trivial, as it is rather long winded and involves at least one unusual trick from advanced calculus.
2. A Simple Substitution

Let $x = \left(\frac{h\nu}{kT}\right)$. Then $dx = \left(\frac{h}{kT}\right)d\nu$. In other words, $\nu = \left(\frac{kT}{h}\right)x$ and $d\nu = \left(\frac{kT}{h}\right)dx$. Eq. 5 becomes

$$E = \pi \left(\frac{2h}{c^2}\right) \left(\frac{kT}{h}\right)^4 \int_0^\infty \frac{x^3dx}{e^x - 1}.$$  

(6)

Using numerical integration and a simple Python program, we can show that the integral on the right hand side of Eq. 6 is approximately equal to 6.493939, and below we show that the integral is exactly equal to $\pi^4/15$. This gives us $E = \sigma T^4$, with

$$\sigma = \frac{2\pi^5 k^4}{15 c^2 h^3} = 5.6704 \times 10^{-5} \text{erg/cm}^2/s/K^4.$$  

(7)

Even before we carry out the difficult integral we can easily see that the energy radiated by a black body varies with the fourth power of the temperature. Let us consider a blue main sequence star of photospheric temperature 30000 K and a red supergiant star of photospheric temperature 3000 K. And say the diameter of the red star is 100 times the diameter of the blue star. The luminosity ($L$) of a star of radius $R$ is equal to the area of the star times the intensity of light emitted per unit area:

$$L = 4\pi R^2 \sigma T^4.$$  

(8)

The blue star radiates most of its energy in the ultraviolet, while the red star radiates most of its energy in the infrared. The red star has a surface area $10^4$ times that of the blue star, but each square meter of the blue star radiates $10^4$ times as much energy (integrated over all frequencies) compared to each square meter of the red star. So the bolometric luminosities of the two stars are equal.

3. Euler’s Gamma Function

We define Euler’s Gamma Function (Hogg & Craig 1970, pp. 99-101) as

$$\Gamma(s) = \int_0^\infty e^{-x}x^{s-1}dx.$$  

(9)

So,
\[ \Gamma(1) = \int_0^\infty e^{-x} \, dx = -e^{-x}\bigg|_0^\infty = 1. \]

\[ \Gamma(2) = \int_0^\infty e^{-x} \, x \, dx. \]

Using integration by parts, let \( u = x \) and \( dv = e^{-x} \, dx \) so that \( du = dx \) and \( v = -e^{-x} \). Then
\[ uv - \int vdu \text{ equals } -xe^{-x}\bigg|_0^\infty + \int_0^\infty e^{-x} \, dx = 0 -0 - e^{-x}\bigg|_0^\infty = 1. \]

Furthermore,
\[ \Gamma(3) = \int_0^\infty e^{-x} \, x^2 \, dx. \]

Using integration by parts, let \( u = x^2 \) and \( dv = e^{-x} \, dx \) so that \( du = 2x \, dx \) and \( v = -e^{-x} \). Then
\[ uv - \int vdu \text{ equals } -x^2e^{-x}\bigg|_0^\infty + \int_0^\infty e^{-x} \, 2x \, dx = -0 + 0 + 2 \cdot 1 = 2. \]

Let us do one more for good measure.
\[ \Gamma(4) = \int_0^\infty e^{-x} \, x^3 \, dx. \]

Using integration by parts, let \( u = x^3 \) and \( dv = e^{-x} \, dx \) so that \( du = 3x^2 \, dx \) and \( v = -e^{-x} \). Then
\[ uv - \int vdu \text{ equals } -x^3e^{-x}\bigg|_0^\infty + \int_0^\infty e^{-x} \, 3x^2 \, dx = -0 + 0 + 3 \cdot 2 \cdot 1 = 6. \]

Thus, each subsequent integral is \( s - 1 \) times an integral we have already done. In summary, for positive integer \( s \)
\[ \Gamma(s) = (s - 1)! \]  

Soon we will need the following (Whittaker & Watson 1935, p. 243):  

\[ \int_0^\infty e^{-bx} x^{s-1} dx = \frac{\Gamma(s)}{b^s}. \]  

Let \( z = bx \), and let us require that \( b > 0 \). Then \( dx = dz/b \) and the integral becomes  

\[ \int_0^\infty e^{-z} \left( \frac{z}{b} \right)^{s-1} \frac{dz}{b} = \int_0^\infty e^{-z} z^{s-1} dz = \frac{\Gamma(s)}{b^s}. \quad (Q.E.D.) \]  

4. The Riemann Zeta Function

Following Whittaker & Watson (1935, pp. 265-6), we define the Riemann Zeta Function as follows:  

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \]  

We can also define a more general Zeta Function thusly:  

\[ \zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(a + n)^s}, \]  

where \( a \) is a constant. For simplicity we shall suppose that \( 0 < a \leq 1 \).

Since  

\[ \sum_{n=0}^{\infty} \frac{1}{(1 + n)^s} = \sum_{n=1}^{\infty} \frac{1}{n^s}, \]  

it is evident that \( \zeta(s, 1) = \zeta(s) \).

Now, using Eq. 11 and \( b = a + n \),
\[
\int_0^\infty x^{s-1} e^{-(a+n)x} \, dx = \frac{\Gamma(s)}{(a+n)^s} .
\] (16)

Whittaker & Watson (1935) have a more formal proof, involving taking the upper limit of the sum in the next equation as \( n \to \infty \), but it boils down to

\[
\sum_{n=0}^\infty \frac{\Gamma(s)}{(a+n)^s} = \sum_{n=0}^\infty \int_0^\infty x^{s-1} e^{-(a+n)x} \, dx
\] (17)

On the right hand side we have an infinite sum of integrals, which can be expressed as follows:

\[
\zeta(s, a) \Gamma(s) = \int_0^\infty x^{s-1} e^{-ax} \left[ 1 + e^{-x} + e^{-2x} + \ldots \right] \, dx
\] (18)

The reader should recall (or you may consult Hodgman 1952, p. 271) that for \( 0 < q < 1 \) the infinite sum of

\[
1 + q + q^2 + \ldots \rightarrow \frac{1}{1-q} .
\] (19)

Letting \( q = e^{-x} \); this leaves us with

\[
\zeta(s, a) \Gamma(s) = \int_0^\infty \frac{x^{s-1} e^{-ax}}{1 - e^{-x}} \, dx .
\] (20)

For \( a = 1 \)

\[
\zeta(s, 1) \Gamma(s) = \zeta(s) \Gamma(s) = \int_0^\infty \frac{x^{s-1} \, dx}{e^x - 1} .
\] (21)

Since \( \Gamma(4) = 3! = 6 \), this means that for \( s = 4 \)

\[
\int_0^\infty x^3 \frac{dx}{e^x - 1} = 6 \sum_{n=1}^\infty \frac{1}{n^4} ,
\] (22)

and we are halfway to finding the exact answer to Eq. 6.
5. An Application of Fourier Series

Following Kaplan (1952, p. 391), let \( f(x) \) be a function that is piecewise very smooth in the interval \(-\pi \leq x \leq \pi\). Then the Fourier series of \( f(x) \) is:

\[
\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),
\]

with

\[
a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx,
\]

\[
a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx,
\]

and

\[
b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx.
\]

Here we wish to determine the Fourier series for \( f(x) = x^2 \). The coefficients \( b_n \) are all zero because sine is an odd function \((f(x) = -f(-x))\), while \( x^2 \) is an even function \((f(x) = f(-x))\), and the integral of this even function times this odd function over the interval of \(-\pi\) to \(\pi\) is zero.

We have

\[
a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \, dx = \frac{x^3}{3\pi}\bigg|_{-\pi}^{\pi} = \frac{2\pi^2}{3}.
\]

Integration by parts allows us to show that

\[
\int x^2 \cos nx \, dx = \frac{x^2}{n} \sin nx + \frac{2x}{n^2} \cos nx - \frac{2}{n^3} \sin nx + C.
\]

Evaluating this result from \( x = -\pi \) to \( +\pi \) for \( n = 1 \) gives \( a_1 = -4 \). For \( n = 2 \), \( a_2 = +1 \). And in general \( a_n = \frac{4}{n^2}(-1)^n \). In the context of Eq. 23 we find
\[ f(x) = x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^2}. \]  

(29)

In Fig. 2 we show the sum of the first 10 terms of the Fourier series given in Eq. 29. Except close to \( x = \pm \pi \) radians the agreement using only 10 cosine terms is quite good.

Here is an interesting intermediate result. For all positive \( n \), \((-1)^n \cos n\pi = +1\). If \( x = \pi \), Eq. 29 gives

\[ \pi^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{1}{n^2}, \]  

(30)

or

\[ \zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}. \]  

(31)

Finally, using the Fourier series for \( f(x) = x^2 \) (Eq. 29), we can determine \( \zeta(4) \) using Parseval’s equation (Kaplan 1952, pp. 412-3):

\[ \frac{1}{\pi} \int_{-\pi}^{\pi} [f(x)]^2 \, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2). \]  

(32)

The left hand side is trivial:

\[ \frac{1}{\pi} \int_{-\pi}^{\pi} x^4 \, dx = \frac{x^5}{5\pi} \bigg|_{-\pi}^{\pi} = \frac{2\pi^4}{5}. \]

This gives us

\[ \frac{2\pi^4}{5} = \frac{2\pi^4}{9} + 16 \sum_{n=1}^{\infty} \frac{1}{n^4}. \]

After a little rearrangement, we have

\[ \zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}, \]  

(33)
which resolves Eqs. 6 and 22.

We thank Eddie Baron for useful discussions regarding fundamentals of radiative transfer discussed in §1.

A. More on the Riemann Zeta Function

If one wishes to calculate $\zeta(6)$ in a similar fashion, one needs the Fourier series

$$f(x) = x^3 = 2 \sum_{n=1}^{\infty} \left[ (-1)^{n+1} \left( \frac{\pi^2}{n} - \frac{6}{n^3} \right) \right] \sin nx . \quad (A1)$$

Then one needs Parseval’s equation, $\zeta(2)$, and $\zeta(4)$, which we have already determined. Using Eqs. 32 and A1 we have

$$\frac{1}{\pi} \int_{-\pi}^{\pi} x^6 \, dx = \sum_{n=1}^{\infty} 4 \left( \frac{\pi^2}{n} - \frac{6}{n^3} \right)^2$$

After dividing both sides by 4, the left hand side is trivial:

$$\frac{1}{4\pi} \int_{-\pi}^{\pi} x^6 \, dx = \frac{x^7}{28\pi} \bigg|_{-\pi}^{\pi} = \frac{\pi^6}{14} .$$

The right hand side becomes

$$\sum_{n=1}^{\infty} \left( \frac{\pi^4}{n^2} - \frac{12\pi^2}{n^4} + \frac{36}{n^6} \right) = \pi^4 \left( \frac{\pi^2}{6} \right) - 12\pi^2 \left( \frac{\pi^4}{90} \right) + 36 \sum_{n=1}^{\infty} \frac{1}{n^6} .$$

Combining all $\pi^6$ terms on the left hand side gives

$$\pi^6 \left( \frac{1}{14} - \frac{1}{6} + \frac{2}{15} \right) = 36 \sum_{n=1}^{\infty} \frac{1}{n^6} .$$

We are almost done.

$$\pi^6 \left( \frac{90 - 210 + 168}{1260} \right) = \pi^6 \left( \frac{48}{1260} \right) = 36 \sum_{n=1}^{\infty} \frac{1}{n^6} .$$
Finally, since $\frac{3}{4}$ of 1260 is equal to 945, we have

$$\zeta(6) = \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}. \tag{A2}$$

The *Wikipedia* article whose URL is given below gives a regression formula for values of the Riemann Zeta Function for an even, positive argument:

$$\zeta(2n) = \left(\frac{1}{n + \frac{1}{2}}\right)^{n-1} \sum_{k=1}^{n-1} \zeta(2k) \zeta(2n - 2k), \quad n > 1. \tag{A3}$$

For $n = 2$,

$$\zeta(4) = \left(\frac{1}{2 + \frac{1}{2}}\right) \left[ \zeta(2) \zeta(4 - 2) \right]$$

$$= \frac{2}{5} \left(\frac{\pi^2}{6}\right) \left(\frac{\pi^2}{6}\right) = \frac{\pi^4}{90},$$

which is the same as the result given in Eq. 33.

For $n = 3$,

$$\zeta(6) = \left(\frac{1}{3 + \frac{1}{2}}\right) \left[ \zeta(2) \zeta(6 - 2) + \zeta(4) \zeta(6 - 4) \right]$$

$$= \left(\frac{1}{3 + \frac{1}{2}}\right) \left[ 2 \left(\frac{\pi^2}{6}\right) \left(\frac{\pi^4}{90}\right) \right]$$

$$= \frac{\pi^6}{945},$$

as given above in Eq. A2. Values of $\zeta(8)$ through $\zeta(34)$ are found in the article whose URL is given in footnote 3.


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REFERENCES


Kaplan, Wilfred 1952, Advanced Calculus, Reading, Massachusetts: Addison Wesley


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Fig. 1.— Examples of the spectral energy distribution of a black body of temperature 6000 K as a function of frequency (top figure) and as a function of wavelength (bottom figure). In the bottom figure we have multiplied the wavelengths in cm by $10^8$ to give wavelengths in Ångströms. Since spectral radiance and spectral energy density only differ by a multiplicative factor, the *shapes* of these curves are the same no matter which set we plot.

Fig. 2.— The sum of first 10 terms of the Fourier series given in Eq. 29 for $f(x) = x^2$. 
Fig. 1. 6000 K black body
Fig. 2.