1. (a) \[ z = \left( \frac{\omega}{n+\frac{1}{2}} \right)^{\frac{3}{2}} \frac{e^{-\frac{3}{2} \frac{\pi}{\omega}/kT}}{1 - e^{-\frac{3}{2} \frac{\pi}{\omega}/kT}} = \frac{e^{-\frac{3}{2} \frac{\pi}{\omega}/kT}}{1 - e^{-\frac{3}{2} \frac{\pi}{\omega}/kT}} \quad \text{if } \omega < 1 \] \[ z = z^N \]

(b) \[ \frac{E}{N} = kT^2 \frac{\partial}{\partial T} \ln z = \frac{1}{2} \frac{\pi}{\omega} + \frac{kT}{e^{\frac{\pi}{\omega}/kT} - 1} \]

(c) \[ \frac{E}{N} = -kT \ln z \quad \Rightarrow \quad \frac{F}{N} = -kT \ln z = \frac{\partial F}{\partial V} = 0 \]

(d) \[ \frac{C_p}{T} = \frac{2 (E + \rho V)}{2T} = \frac{\partial F}{\partial V} = \frac{C_V}{T} \]

(e) \[ \frac{E}{N} = \frac{1}{2} \frac{\pi}{\omega} + \frac{kT}{e^{\frac{\pi}{\omega}/kT} - 1} \left[ 1 + \frac{1}{2} \left( \frac{kT}{\pi} \right)^2 + \ldots \right]^{-1} \]

\[ = \frac{1}{2} \frac{\pi}{\omega} + \frac{kT}{e^{\frac{\pi}{\omega}/kT} - 1} \frac{1}{1 - \frac{1}{2} \left( \frac{kT}{\pi} \right)^2 + \ldots} \]

\[ = \left[ kT \left[ 1 + O \left( \frac{1}{2} \left( \frac{kT}{\pi} \right)^2 \right) \right] \right] \]

2. (a) \[ dS = \left( \frac{\partial S}{\partial V} \right)_T dV + \left( \frac{\partial S}{\partial T} \right)_V dT = \left( \frac{\partial S}{\partial V} \right)_T dV + \frac{C_V}{T} dT \]

(b) \[ dF = -S dT - P dV \Rightarrow \left( \frac{\partial S}{\partial V} \right)_T = -S \frac{\partial F}{\partial V} = -P \]

\[ \text{so} \quad \left( \frac{\partial S}{\partial V} \right)_T = -S \frac{\partial F}{\partial V} = -S \frac{\partial F}{\partial V} = \left( \frac{\partial P}{\partial T} \right)_V \]

(c) \[ dE = T dS - P dV = T \left[ \left( \frac{\partial P}{\partial V} \right)_T dV + \frac{C_V}{T} dT \right] - P dV \]

\[ = \left[ T \left( \frac{\partial P}{\partial T} \right)_V - P \right] dV + C_V dT \]

(d) \[ \left( \frac{\partial E}{\partial V} \right)_T = T \left( \frac{\partial P}{\partial T} \right)_V - P \]

\[ P + \frac{a}{v^2} = \frac{NK}{V-N} b = \frac{kT}{V-b} \Rightarrow \left( \frac{\partial P}{\partial T} \right)_V = \frac{k}{v-b} \]

\[ \left( \frac{\partial E}{\partial V} \right)_T = \frac{kT}{v-b} - \left( \frac{kT}{v-b} - \frac{a}{v^2} \right) = \frac{a}{v^2} \]

\[ \text{Then} \quad E = E_0(T) - a \frac{N}{v} \quad , \quad v = \frac{V}{N} \]

\[ \text{Note: this is a skeleton solution - not a model!} \]
3. In (a) \( \ell (b) = 1 \), \( \ln (i) = 0 \) and \( \ln n \to 0 \) as \( n \to \infty \)

(a) \( \langle n_k \rangle \) and \( 1 - \langle n_k \rangle = 0 \) or \( 1 \) in ground state, so \( S \to 0 \).

(b) \( \langle n_k \rangle = 0 \) for excited states, \( \langle n_k \rangle = N \) for lowest state

\[
N \ln N - (1 + N) \ln (1 + N) = N \ln N - (1 + N) \ln (N(1 + \frac{1}{N}))
\]
\[
= N \ln N - (1 + N) \ln N - (1 + N) \left( \frac{1}{N} + O \left( \frac{1}{N^2} \right) \right)
\]
\[
= -\ln N + 1 + O \left( \frac{1}{N} \right)
\]
\[
\Rightarrow \frac{S}{N} = -\frac{\ln N}{N} + O \left( \frac{1}{N} \right) \to 0 \text{ as } N \to \infty
\]

(c) & (d) We have essentially

\[
\frac{2}{\alpha n} \left[ -n \ln n + (1 + n) \ln (1 + n) - \gamma n - \beta n E \right] = 0
\]
\[
\Rightarrow 0 = -\ln n + n \cdot \frac{1}{n} + (1 + n) \ln (1 + n) - \gamma n - \beta n E
\]
\[
= -\ln n + 1 + \ln (1 + n) + 1 - \gamma - \beta E
\]
\[
= \ln \left( \frac{1 + n}{n} \right) - \beta - \beta E
\]
\[
\Rightarrow \ln \left( \frac{n^{-1} + 1}{n} \right) = \gamma + \beta E
\]
\[
\Rightarrow n^{-1} + 1 = e^{\gamma + \beta E}
\]
\[
\Rightarrow n^{-1} = e^{\beta E + \gamma} - 1
\]
\[
\Rightarrow \gamma = \frac{1}{e^{\beta E + \gamma} - 1}
\]

If we let \( \gamma = -\beta \mu \), we have the form

\[
\eta_k = \frac{1}{e^{\beta (E_k - \mu)} + 1}
\]

of the Fermi-Dirac and Bose-Einstein distribution functions.
\[ \frac{dG}{dt} = \frac{1}{c} \int_0^t dt' \frac{dG_c}{dt'} = \frac{1}{c} (G(t) - G(0)) \to 0 \text{ as } t \to \infty \]

\[ \frac{dG}{dt} = \frac{d}{dt} \sum \dot{q}_i \dot{y}_i = \sum \dot{q}_i \dot{y}_i + \sum \ddot{q}_i \ddot{y}_i \]

\[ \sum \dot{q}_i \dot{y}_i + \sum \dot{q}_i \dot{y}_i = 0 \text{ or } \sum \dot{q}_i \dot{y}_i = -\sum \dot{q}_i \dot{y}_i \]

Since \( u_i = \dot{q}_i \) and \( \ddot{y}_i = \ddot{F}_i \)

\[ \sum \dot{q}_i \dot{y}_i = \sum \left( \ddot{F}_i - F_i \right) = \sum \left( r \left( -\frac{G m M}{r^2} \right) \right) = \sum U_{\text{planet}} \]

\[ \Rightarrow \langle U_{\text{planet}} \rangle = -2 \langle K_{\text{planet}} \rangle \]

\[ 2 \langle \frac{1}{2} \dot{y} \dot{y} \rangle = G \langle \frac{\dot{y} \dot{y}}{r} \rangle \Rightarrow \dot{y} \dot{y} = \frac{G M(r)}{r} \]

\[ \Rightarrow M(r) = \frac{\dot{y} \dot{y}}{G} \]

\[ \int S \ddot{r} \cdot dS = \int V \ddot{r} \cdot d^3r = \int V \left( \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} \right) d^3r = 3 \int d^3r = 3V \]

\[ \Rightarrow N \langle \ddot{r} \cdot \ddot{F}_{\text{walls}} \rangle = -P \Rightarrow N \text{ and this is } \sum \langle q_i \dot{F}_{i, \text{walls}} \rangle \]

\[ \begin{align*}
\sum \langle q_i \dot{F}_{i, \text{walls}} \rangle &= \sum \langle q_i \dot{F}_{i, \text{walls}} \rangle + 3PV \\
\end{align*} \]

\[ \frac{\text{d}U}{\text{d} \lambda} = \frac{\partial U}{\partial \lambda} \frac{\partial \lambda}{\partial \lambda} + \ldots \]

\[ = \frac{\partial U}{\partial \lambda} \frac{\partial \lambda}{\partial \lambda} + \ldots \]

\[ \text{and this also } = \frac{\partial^2}{\partial \lambda^2} \left( \lambda^n U(q_1, \ldots) \right) = n \lambda^n U(q_1, \ldots) \]

Set \( \lambda = 1 \):

\[ \frac{\partial U}{\partial q_1} \frac{\partial q_1}{\partial q_1} + \frac{\partial U}{\partial q_2} \frac{\partial q_2}{\partial q_2} + \ldots = n U(q_1, q_2, \ldots) \]

Then \( \sum \langle q_i \dot{F}_{i, \text{walls}} \rangle = -n \langle U \rangle \), with \( \langle K \rangle = \frac{1}{2} m \langle \dot{y} \dot{y} \rangle \)

\[ 2 \langle K \rangle = n \langle U \rangle + PV \Rightarrow \langle K \rangle = \frac{n}{2} \langle U \rangle + \frac{3}{2} PV \]

\[ \Rightarrow 0 = \langle U \rangle \Rightarrow \langle K \rangle = \frac{3}{2} PV \]

\( \Rightarrow P = 0, n = -1 \) result like (b); \( \langle U \rangle = 0 \Rightarrow \langle K \rangle = \frac{3}{2} PV \]

\[ -\sum \langle q_i \dot{F}_{i, \text{walls}} \rangle = -\sum \langle q_i \dot{q}_i \rangle = +\sum \langle q_i \dot{q}_i \rangle = 3NkT \]

\[ \Rightarrow \text{from (d), } 3NkT = 3PV \Rightarrow PV = NkT \]