Physics 607 Exam 2

Please be well-organized, and show all significant steps clearly in all problems.

Do all your work on the blank sheets provided, writing your name clearly. (You may keep this exam.)

The variables have their usual meanings: \( E = \) energy, \( S = \) entropy, \( V = \) volume, \( N = \) number of particles, \( T = \) temperature, \( P = \) pressure, \( \mu = \) chemical potential, \( B = \) applied magnetic field, \( C_V = \) heat capacity at constant volume, \( C_P = \) heat capacity at constant pressure, \( F = \) Helmholtz free energy, \( G = \) Gibbs free energy, \( k = \) Boltzmann constant, \( h = \) Planck constant, \( c = \) speed of light. Also, \( \langle \cdots \rangle \) represents an average.

In working these problems, you may assume (when appropriate) the following:

\[
\int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi} \quad \int_{-\infty}^{\infty} x^2 e^{-x^2} \, dx = \frac{1}{2} \sqrt{\pi} \quad \int_{0}^{\infty} e^{-x} x^{z-1} \, dx = \Gamma(z) \quad \Gamma(n+1) = n!
\]

\[
\langle E \rangle = kT^2 \frac{\partial}{\partial T} \ln Z \
Z = \sum_{r} e^{-\beta Er} \
\Omega = -kT \ln Z \
Z = \sum_{N} e^{-\beta E_{Nr}} e^{-\gamma N} 
\]

\[
F = -kT \ln Z \
\Omega = -kT \ln Z \
Z = \frac{1}{N!} \int dq \, dp \, e^{-H(p,q)/kT} 
\text{Tr} \left( \frac{\partial H}{\partial x} \right) = \delta_{ij} kT
\]
1. (25) Here you are asked to prove the **Bohr–van Leeuwen theorem**: The magnetic susceptibility is identically zero in classical mechanics.*

We are concerned with a system of charged particles (not dipoles), obeying **classical mechanics and classical statistics**.

The system is assumed to be the kind that we normally consider, in thermal equilibrium, with a uniform magnetic field, in the thermodynamic limit, with no boundary effects, no quantum effects of any kind, etc.

It has a classical Hamiltonian, which in the presence of a magnetic field

\[
\vec{B} = \nabla \times \vec{A}
\]

is just (with interactions between particles regarded as irrelevant and ignored)

\[
H = \sum_i \frac{1}{2m} \vec{P}_i^2 = \sum_i \frac{1}{2m} \left( \vec{p}_i - \frac{q}{c} \vec{A}(\vec{r}_i) \right)^2
\]

where \(\vec{P}\) is the mechanical momentum and \(\vec{p}\) is the canonical momentum.

Recall that \(M = -\left( \frac{\partial F}{\partial B} \right)_T\) and take note of the equations on the first page of this exam.

**Demonstrate that the magnetic susceptibility of the classical system above is identically zero.**

(Please give a clear precise demonstration, explaining and emphasizing the essential points.)

* People have explored various nuances, but here we are concerned with a normal classical system of the kind described above.

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**Hendrika Johanna van Leeuwen, Dutch physicist**  
(1887 – 1974)

Her doctoral thesis explained why magnetism is essentially a quantum mechanical effect -- a result now referred to as the Bohr–van Leeuwen theorem.

[https://en.wikipedia.org/wiki/Hendrika_Johanna_van_Leeuwen]
2. In class, for an ideal quantum gas consisting of $N$ fermions we obtained

$$C_V = \frac{1}{3} \pi^2 k^2 \rho(\epsilon_F) T$$

where $\epsilon_F$ is the Fermi energy. Here we are concerned with a (nonrelativistic) free electron model in 2 dimensions. (This could be a model of a thin metallic film.)

(a) (5) Recall that the area in momentum space per allowed momentum is $\hbar^2 / A$ (where $A$ is the spatial area occupied by the gas), with periodic boundary conditions.

Obtain the 2-dimensional density of states as a function of energy, $\rho(\epsilon)$. 

(b) (5) Then show that

$$\epsilon_F = \text{constant } \times \frac{N}{A}$$

where you will obtain the constant.

(c) (5) Show that

$$\frac{C_V}{N} = \text{constant}^\prime \times \frac{kT}{\epsilon_F}$$

where you will obtain constant’.

(d) (5) Show that

$$\frac{E(0)}{N} = \text{constant}^\prime\prime \times \epsilon_F$$

where you will obtain constant’’. Here $E(0)$ is the energy of the system at $T = 0$.

Does it make sense that an average electron has this much energy?
3. (25) The velocity of sound is obtained from the adiabatic compressibility*

\[
\kappa_S \equiv -\frac{1}{V} \left( \frac{\partial V}{\partial P} \right)_S .
\]

Calculate \(\kappa_S\) for an ideal (quantum) Fermi gas, in terms of the functions

\[
f_n(z) = \sum_{\ell=1}^{\infty} (-1)^{\ell-1} \frac{z^\ell}{\ell^n} = z - \frac{z^2}{2^n} + \frac{z^3}{3^n} - ...
\]

where \(z\) is the fugacity in the textbook’s notation. (You may instead use our notation in class for the fugacity if you wish, with \(z \to \lambda\) throughout.)

You may assume the equation**

\[
P_v^{5/3} = \text{constant for an adiabatic process} \quad , \quad v \equiv V / N
\]

or

\[
S = N k \left( \frac{5}{2} \frac{f_{5/2}(z)}{f_{3/2}(z)} - \ln z \right)
\]

(or both, but both are not required).

You may also use

\[
\frac{P}{kT} = \frac{1}{\lambda_{th}^3} f_{5/2}(z) \quad , \quad \frac{N}{V} = \frac{1}{\lambda_{th}^3} f_{3/2}(z) \quad , \quad \frac{\partial f_n(z)}{\partial z} = \frac{1}{z} f_{n-1}(z) \quad , \quad \text{with} \quad \lambda_{th} = \frac{h}{(2\pi mkT)^{1/2}}
\]

although you do not need all of these equations.

**Give your answer in terms of only the following quantities:**

the number density \(n \equiv \frac{N}{V}\), the Boltzmann constant \(k\), the temperature \(T\), \(f_{3/2}(z)\), and \(f_{5/2}(z)\).

(I.e., we want a convenient form in which quantities like \(\lambda_{th}\), the particle mass, Planck's constant, the pressure, etc. do not appear.)

*Not the isothermal compressibility \(\kappa_T\), since there is ordinarily not enough time during one period of oscillation to achieve thermal equilibrium. The velocity of sound is given by

\[
v_{\text{sound}} = \sqrt{\frac{\partial P}{\partial \rho_m}}_S = \sqrt{\kappa_S^{-1}} = \sqrt{\frac{\text{stiffness}}{\text{mass density}}} .
\]

**This equation is also true for a classical ideal gas of particles with no internal excitations, but for the quantum ideal gas considered here \(\gamma \equiv \frac{C_P}{C_V} = \frac{5}{3} f_{5/2}(z) f_{1/2}(z) \frac{5}{3} \left( f_{3/2}(z) \right)^2 \neq \frac{5}{3}\). I.e., for a quantum ideal gas, it is no longer true that \(P_v^\gamma = \text{constant during an adiabatic process}\).
4. The Navier-Stokes equations of fluid dynamics are more powerful than the earlier Euler equations, because they include viscosity and thermal conductivity. We therefore need to use the Navier-Stokes equations if we are to calculate both the wave velocity and the damping constant for a sound wave moving through a viscous fluid, like air or water – which we now set out to do.

(a) (4) Consider a density $f$ of some scalar or vector (or tensor) physical quantity which is being carried by a classical fluid in 3 dimensions with local velocity $\mathbf{v}$. Using a sketch to illustrate your arguments, give a simple but complete demonstration that

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f = \left[ \frac{\partial f}{\partial t} \right]_{\text{sources}}$$

where the quantity on the right-hand side has an obvious meaning.

(b) (4) Now write down the Navier-Stokes equations for (1) the mass density $\rho$ and (2) the momentum density $\rho \mathbf{v}$, if mass is conserved but the source terms in (2) are a pressure gradient and a shear viscosity $\eta$:

$$\frac{\partial (\rho \mathbf{v})}{\partial t} = -\mathbf{v} \cdot \nabla P + \eta \nabla^2 \mathbf{v} + \frac{1}{3} \eta \mathbf{v} \cdot \nabla (\mathbf{v} \cdot \mathbf{v})$$

You may neglect a term which is second-order in $\mathbf{v}$ since we linearize below.

(c) (4) Using the equations of part (b), obtain an equation involving $\frac{\partial^2 \rho}{\partial t^2}$, $\nabla^2 P$, and $\nabla^2 \left( \mathbf{v} \cdot \mathbf{v} \right)$.

(d) (4) Note that $\nabla^2 P$ can be written as $\mathbf{v} \cdot \nabla P$ and that $\frac{\partial P}{\partial x} = \frac{\partial P}{\partial \rho} \frac{\partial \rho}{\partial x}$. Also assume that

$$\left( \frac{\partial \rho}{\partial P} \right)_s = \rho \kappa_s$$

is approximately constant, where $\kappa_s$ is the adiabatic compressibility. Then replace $\nabla^2 P$ by an expression involving $\nabla^2 \rho$ in the equation of part (c).

(e) (4) Replace $\mathbf{v} \cdot \mathbf{v}$ by an expression involving $\frac{\partial \rho}{\partial t}$, so that you finally obtain an equation involving just $\frac{\partial^2 \rho}{\partial t^2}$, $\nabla^2 \rho$, $\nabla^2 \frac{\partial \rho}{\partial t}$, and constants. Assume that $\mathbf{v} \cdot \nabla \rho$ is small compared to $\frac{\partial \rho}{\partial t}$ and that $\nabla^2 \left( \frac{1}{\rho} \frac{\partial \rho}{\partial t} \right) = \frac{1}{\rho} \nabla^2 \left( \frac{\partial \rho}{\partial t} \right)$.

(f) (10) Now assume a wave with the form $\rho = \rho_0 + \delta \rho$, $\delta \rho = a e^{i k \mathbf{r}} e^{-i \omega t}$, where $\rho_0$ is a constant and $\omega$ is complex: $\omega = \omega_r \pm i \omega_i$, where you will choose the sign corresponding to damping of the wave (rather than unphysical growth with time).

Calculate the sound velocity, $c = \text{real part of} \frac{\partial \omega}{\partial k}$, and the damping constant $\gamma$ which is defined by the factor $e^{-\gamma t/2}$ in the solution for $\delta \rho$. The results should be given in terms of the density $\rho_0$, the compressibility $\kappa_s$, the viscosity $\eta$, and the wavenumber $k$. (Assume that $k$ can be taken to be real in this context, and that $\omega_i$ is small compared to $\omega_r$.)