(1a) Show that if a Lorentz tensor vanishes in one Lorentz frame, it vanishes in all Lorentz frames.

(1b) Show that if a Lorentz tensor $A_{\mu\nu}$ is antisymmetric in one Lorentz frame, it is antisymmetric in all Lorentz frames.

(1c) You are given a Lorentz tensor $K^{\mu\nu}$. How would you test whether it is the outer product of two vectors, so that $K^{\mu\nu} = A^{\mu} B^{\nu}$? Can you express the test in a coordinate-free language?

(1d) Prove, making use of the general expression for the Lorentz transformation of a $(p,q)$ tensor given in eqn (2.46) in the notes, that the $(1,1)$ Kronecker tensor $\delta^{\mu}_{\nu}$ is an invariant tensor.

(2) Suppose that $U^\mu$ and $V^\mu$ are any pair of Lorentz 4-vectors, and suppose that the equation $U^\mu = W^{\mu\nu} V^\nu$ holds in all Lorentz frames. Prove that $W^{\mu\nu}$ must be a Lorentz tensor. This is an example of the so-called “Quotient Rule.”

(3) Calculate the transformation of the Minkowski metric $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$ under

$$x'^\mu = \frac{b^2}{\Omega^2} (x^\mu + \alpha^\mu), \quad \text{with} \quad \Omega^2 = \eta_{\mu\nu} (x^\mu + \alpha^\mu)(x^\nu + \alpha^\nu),$$

where $b$ and $\alpha^\mu$ are constants. Show that they have the property of preserving the null separation condition $ds'^2 = 0$, although they do not leave a non-vanishing $ds^2$ invariant. These transformations, together with the Poincaré transformations, generate the $5 + 10 = 15$ parameter conformal group that leaves $ds^2 = 0$ invariant. This would be a symmetry in physics if the only particles that existed in nature were massless.

Due in class on Wednesday 16th September