Problem Sheet 5

(1a) Show that the energy-momentum tensor for the electromagnetic field is tracefree, i.e. $T^\mu_\mu = 0$. What would happen, in a spacetime dimension $d \neq 4$? (Assume the expression for $T_{\mu\nu}$ in terms of the antisymmetric tensor $F_{\mu\nu}$ is unchanged in all dimensions.)

(1b) Calculate the components of $\ast F_{\mu\nu}$ in terms of $\vec{E}$ and $\vec{B}$. Show that $\ast F_{\mu\nu}$ can be obtained from $F_{\mu\nu}$ by sending $\vec{E} \to -\vec{B}$ and $\vec{B} \to \vec{E}$.

(2a) Show that the energy-momentum tensor for the electromagnetic field can be written as

$$T_{\mu\nu} = \frac{1}{8\pi} (F_{\mu\rho} F^\rho_\nu + \ast F_{\mu\rho} \ast F^\rho_\nu).$$

(2b) Show that the energy-momentum tensor for the electromagnetic field satisfies

$$T_{\mu\rho} T^{\nu\rho} = \frac{1}{(8\pi)^2} \left[ (E^2 - B^2)^2 + (2\vec{E} \cdot \vec{B})^2 \right] \delta_{\nu}^\mu.$$

(Note: It is rather non-trivial that $T_{\mu\rho} T^{\nu\rho}$ is proportional to $\delta_{\nu}^\mu$. Make sure that you prove this, and don’t simply assume it!)

(3) Derive the equations of motion (Euler-Lagrange equations) that follow from the Lagrangian density

$$\mathcal{L} = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} - \frac{m^2}{8\pi} A^\mu A_\mu + J^\mu A_\mu$$

where $m$ is a constant, and $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$. This theory is a massive generalisation of Maxwell’s theory, known as the Proca theory. Find the solution for the scalar potential $\phi \equiv A^0$ describing a “point charge” $q$ located at the origin. (i.e. find the analogue in Proca theory of the usual static, spherically symmetric point charge in ordinary electrodynamics.)

Due on Wednesday 22nd October