(1) Many properties of the Bessel functions can be derived from a generating function.
This defines the $J_n(x)$ (for $n$ an integer) as follows:

$$G(x,t) \equiv e^{\frac{1}{2}x(t^{1}t^{-1})} = \sum_{n=-\infty}^{\infty} t^n J_n(x)$$

All the following problems should be solved using this definition.

(1a) Show that $J_n(x)$ as defined above does indeed satisfy Bessel's equation $x^2 J''_n + x J'_n + (x^2 - n^2) J_n = 0$. (The proof is analogous to the one given in the lectures for the Legendre polynomials.)

(1b) Show that $J_{n-1}(x) - J_{n+1}(x) = 2J_n'(x)$.

(1c) Show that $J_{n-1}(x) + J_{n+1}(x) = (2n/x)J_n(x)$.

(1d) Show that $J_{-n}(x) = (-1)^n J_n(x)$.

(2) A hollow right cylinder of radius $a$ has its axis along the $z$ axis, and it has end caps at $z = 0$ and $z = h$. The potential on the end caps is zero, and on the cyclindrical surface at $\rho = a$ it is given by $\phi(a, \varphi, z) = V(\varphi, z)$, where $V(\varphi, z)$ is a given function. By making the appropriate separation of variables in cylindrical coordinates, find a series solution for the potential everywhere inside the cylinder. (Obtain expressions, as integrals involving $V(\varphi, z)$, for the coefficients in the series solution.)

(3a) By taking $t \rightarrow i e^{i \varphi}$, $x \rightarrow k \rho$, use the generating function for Bessel functions in question (1) to prove that

$$e^{ik\rho \cos \varphi} = \sum_{n=-\infty}^{\infty} i^n e^{in\varphi} J_n(k\rho).$$

(3b) Show (using an elementary calculation in Cartesian coordinates) that $\phi = e^{i(z+ix)}$ satisfies Laplace's equation (now, in this part of the problem, $x$ and $z$ are Cartesian coordinates). Writing $\phi$ in cylindrical polars, show using the result in part (3a) that $\phi$ is a special case of the general form of the solutions discussed in section 5 of the lectures.
(4) The goal in this problem is to prove that
\[
\int_0^\infty J_m(k\rho)J_m(k\rho')dk = \frac{1}{\rho} \delta(\rho - \rho')
\] (1)

Do this by starting from the well-known result in 2D Cartesian coordinates that
\[
\frac{1}{4\pi^2} \int_{-\infty}^\infty dk_1 \int_{-\infty}^\infty dk_2 e^{i(k_1(x-x')+k_2(y-y'))} = \delta(x-x')\delta(y-y')
\] (2)

Note that in polar coordinates, the right-hand side becomes \(\frac{1}{\rho} \delta(\rho - \rho') \delta(\varphi - \varphi')\). Using the relation (see qu. (3a) above) \(e^{ik\rho\cos(\varphi-\alpha)} = \sum_{m=-\infty}^{\infty} i^m J_m(k\rho) e^{im(\varphi-\alpha)}\), re-express (2) in terms of polar variables. (In particular, change to polar coordinates in \(k\)-space for the integrations.) Making use of standard results from Fourier analysis, show that (1) holds.

(5) Consider the Dirichlet Green function in the unbounded space between planes at \(z = 0\) and \(z = h\). Using cylindrical polar coordinates, show that it can be written as
\[
G(\vec{r}, \vec{r}') = 2 \sum_{m=-\infty}^{\infty} \int_0^\infty dk e^{im(\varphi-\varphi')} J_m(k\rho)J_m(k\rho') \frac{\sinh(kz_<)\sinh(k(h - z_>))}{\sinh(kh)}
\]

\((z_<\text{ and } z_>\text{ denote respectively the smaller and larger of } z \text{ and } z').\)

Due on Wednesday 25th March