± PSS 1.2 Unit Conversions

Learning Goal:
To practice Problem-Solving Strategy 1.2 Unit Conversions.
A gallon of water in the United States weighs about 8.33 lb. In other words, the density of water is 8.33 lb/gal. What is the density of water in kg/m³? What is the density of water in g/cm³?

Problem-Solving Strategy: Unit conversions

IDENTIFY the relevant concepts:
In most cases, you're best off using the fundamental SI units (meters, kilograms, seconds) within a problem. If you need the answer to be in a different set of units, wait until the end of the problem to make the conversion.

SET UP the problem and EXECUTE the solution as follows:
Units are multiplied and divided just like ordinary algebraic symbols. This gives us an easy way to convert a quantity from one set of units to another. The key idea is to express the same physical quantity in two different units and form an equality. For example, since \( \frac{1 \text{ min}}{60 \text{ sec}} \) equals 1.

EVALUATE your answer:
If you do your units conversion correctly, unwanted units will cancel. Finally, check whether your answer is reasonable. If you have converted to a smaller unit, for example, the number representing the quantity should be larger.

IDENTIFY the relevant concepts
The physical property of density is given by mass/volume. The SI unit for mass is the kilogram (kg) and for volume it is the cubic meter (m³). Therefore, density should be given in units of kg/m³.

SET UP the problem and EXECUTE the solution as follows

Part A
Calculate the density of water in kg/m³.

Express your answer in kilograms per cubic meter using three significant figures.

Hint 1. Find the conversion factor between pounds and kilograms
Which of the following expressions is the correct conversion factor needed when converting pounds into kilograms?

ANSWER:
When you convert pounds into kilograms, you need to multiply by a conversion factor that has pounds in the denominator and equals 1. For example, you can start from the equality \(1.00 \text{ kg} = 2.20 \text{ lb}\) and derive the conversion factor

\[
\frac{1.00 \text{ kg}}{2.20 \text{ lb}} = 1.
\]

**Hint 2. Find the conversion factor between gallons and cubic meters**

Which of the following expressions is the correct conversion factor needed when converting gallons into cubic meters?

**ANSWER:**

When you convert gallons into cubic meters, you need to multiply by a conversion factor that has gallons in the denominator and equals 1. For example, you can start from the equality \(1.00 \text{ m}^3 = 264 \text{ gal}\) and derive the conversion factor

\[
\frac{1.00 \text{ m}^3}{264 \text{ gal}} = 1.
\]

**ANSWER:**

\[
8.33 \text{ lb/gal} = 1000 \text{ kg/m}^3
\]

If you were to complete the conversion and find the equivalent density in \(\text{kg/m}^3\) of 8.33 \(\text{ lb/gal}\), you would write

\[
8.33 \frac{\text{lb}}{\text{gal}} = 8.33 \frac{\text{lb}}{\text{gal}} \times \frac{1.00 \text{ kg}}{2.20 \text{ lb}} \times \frac{264 \text{ gal}}{1.00 \text{ m}^3} = 1000 \frac{\text{kg}}{\text{m}^3}
\]

**Part B**

Now, convert the density of water to \(\text{g/cm}^3\). 

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Express your answer in grams per cubic centimeter using three significant figures.

**Hint 1. Find the conversion factor between cubic meters and cubic centimeters**

Which of the following is the correct relationship between cubic meters and cubic centimeters?

**ANSWER:**

- $1 \text{ m}^3 = 10^6 \text{ cm}^3$
- $1 \text{ m}^3 = 10^4 \text{ cm}^3$
- $1 \text{ m}^3 = 10^2 \text{ cm}^3$
- $1 \text{ m}^3 = 10^{-2} \text{ cm}^3$
- $1 \text{ m}^3 = 10^{-4} \text{ cm}^3$
- $1 \text{ m}^3 = 10^{-6} \text{ cm}^3$

The length conversion

$$\frac{1 \text{ m}}{100 \text{ cm}} = 1$$

becomes the volume conversion

$$\left( \frac{1 \text{ m}}{100 \text{ cm}} \right)^3 = \frac{1 \text{ m}^3}{10^6 \text{ cm}^3} = 1.$$

**ANSWER:**

$$1000 \ \text{kg/m}^3 = 1.00 \ \text{g/cm}^3$$

If you had 1 cm$^3$ of water (about the size of a sugar cube), it would have a mass of 1 g.

**EVALUATE your answer**

**Part C**

The same physical quantity, such as density, can be reported using different units. Above, you found that water has a density of 1000 kg/m$^3 = 1$ g/cm$^3$. Because the density of water must be the same regardless of what units you use to measure it, you can conclude that an object whose density is 1 kg/m$^3$ must be less dense than water. In other words, 1 kg/m$^3$ is less than 1 g/cm$^3$.

If you had three different objects with densities of 1 kg/m$^3$, 1 g/cm$^3$, and 1 kg/mm$^3$, which object would be the most dense?

Rank the given densities from most to least dense. To rank items as equivalent, overlap them.

**Hint 1. How to approach the problem**

If all these densities were given in the same units, you could easily compare the objects to identify the most dense and least dense. Since the units given are all different, convert all the densities to a common set of units, such as kg/m$^3$, before making your comparison.
These three densities are readily compared if they are converted to the SI unit of density, kg/m³:

\[
\begin{align*}
1 \text{ kg/m}^3 & = 10^0 \text{ kg/m}^3 \\
1 \text{ kg/m}^3 & = 1 \text{ kg/m}^3 \\
1 \text{ g/m}^3 & = 10^{-3} \text{ kg/m}^3
\end{align*}
\]

Converting Units: The Magic of 1

Learning Goal:

To learn how to change units of physical quantities.

Quantities with physical dimensions like length or time must be measured with respect to a unit, a standard for quantities with this dimension. For example, length can be measured in units of meters or feet, time in seconds or years, and velocity in meters per second.

When solving problems in physics, it is necessary to use a consistent system of units such as the International System (abbreviated SI, for the French Système International) or the more cumbersome English system. In the SI system, which is the preferred system in physics, mass is measured in kilograms, time in seconds, and length in meters. The necessity of using consistent units in a problem often forces you to convert some units from the given system into the system that you want to use for the problem.

The key to unit conversion is to multiply (or divide) by a ratio of different units that equals one. This works because multiplying any quantity by one doesn't change it. To illustrate with length, if you know that \(1 \text{ inch} = 2.54 \text{ cm}\), you can write

\[
1 = \frac{2.54 \text{ cm}}{1 \text{ inch}}
\]

To convert inches to centimeters, you can multiply the number of inches times this fraction (since it equals one), cancel the inch unit in the denominator with the inch unit in the given length, and come up with a value for the length in centimeters. To convert centimeters to inches, you can divide by this ratio and cancel the centimeters.

For all parts, notice that the units are already written after the answer box; don't try to write them in your answer also.

Part A

How many centimeters are there in a length 60.7 inches?
Express your answer in centimeters to three significant figures.

\[ 2.54d_1 = 154 \text{ cm} \]

Sometimes you will need to change units twice to get the final unit that you want. Suppose that you know how to convert from centimeters to inches and from inches to feet. By doing both, in order, you can convert from centimeters to feet.

**Part B**

Suppose that a particular artillery piece has a range \( R = 9590 \text{ yards} \). Find its range in miles. Use the facts that \( 1 \text{ mile} = 5280 \text{ ft} \) and \( 3 \text{ ft} = 1 \text{ yard} \).

Express your answer in miles to three significant figures.

**Hint 1. Convert yards to feet**

The first step in this problem is to convert from yards to feet, because you know how to then convert feet into miles. Convert 9590 \( \text{ yards} \) into feet. Use

\[ \frac{1}{1 \text{ yard}} = \frac{3 \text{ ft}}{1 \text{ yard}} \]

Express your answer in feet to three significant figures.

\[ 3R = 2.88 \times 10^4 \text{ ft} \]

Now, you can convert your unrounded answer 28770 \( \text{ ft} \) to miles by using

\[ \frac{1}{5280 \text{ ft}} = \frac{1 \text{ mile}}{5280 \text{ ft}} \]

\[ 9590 \text{ yards} = \frac{R \cdot 3}{5280} = 5.45 \text{ miles} \]

Often speed is given in miles per hour (mph), but in physics you will almost always work in SI units. Therefore, you must convert mph to meters per second (m/s).

**Part C**

What is the speed of a car going \( v = 1000 \text{ mph} \) in SI units? Notice that you will need to change from miles to meters and from hours to seconds. You can do each conversion separately. Use the facts that \( 1 \text{ mile} = 1609 \text{ m} \) and \( 1 \text{ hour} = 3600 \text{ s} \).

Express your answer in meters per second to four significant figures.

**Hint 1. Convert miles to meters**

In converting 1000 mph into meters per second, you will need to multiply by
When you do this, the miles will cancel to leave you with a value in meters per hour. You can then finish the conversion. What is \( v = 1.000 \text{ mph} \) in meters per hour?

Express your answer in meters per hour to four significant figures.

ANSWER:

\[
\begin{align*}
v &= 1609 \ \text{m/hour}
\end{align*}
\]

**Hint 2. Convert hours to seconds**

Which of the following would you multiply \( 1609 \text{ m/hour} \) by to convert it into meters per second (\( \text{m/s} \))?

ANSWER:

\[
\begin{align*}
\text{Answer: } 3600 \text{ s}
\end{align*}
\]

Notice that by equating the two values for \( v \), you get \( 1.000 \text{ mph} = 0.4469 \text{ m/s} \). It might be valuable to remember this, as you may frequently need to convert from miles per hour into more useful SI units. By remembering this relationship in the future, you can reduce this task to a single conversion.

**Significant Figures**

**Part A**

To seven significant figures, the mass of a proton is \( 1.672523 \times 10^{-27} \text{ kg} \). Which of the following choices demonstrates correct rounding?

Check all that apply.

ANSWER:

\[
\begin{align*}
\text{Answer: } 1.67 \times 10^{-27} \text{ kg}
\end{align*}
\]

The number \( 1.672 \times 10^{-27} \text{ kg} \) is incorrect because when we round to four significant figures we get 1.673, not 1.672. Similarly, \( 1.67263 \times 10^{-27} \text{ kg} \) is incorrect because when we round to six significant figures we get 1.67262, not 1.67263.
Part B

To eight significant figures, Avogadro's constant is $6.0221357 \times 10^{23} \text{ mol}^{-1}$. Which of the following choices demonstrates correct rounding?

Check all that apply.

**ANSWER:**

- $6.022 \times 10^{23} \text{ mol}^{-1}$
- $6.0 \times 10^{23} \text{ mol}^{-1}$
- $6.02214 \times 10^{23} \text{ mol}^{-1}$

All these options are correct; they represent different levels of precision, even though the numerical value is the same.

± Vector Addition

Consider the following three vectors:

\[ \vec{A} = (2, -1, 1) \]
\[ \vec{B} = (3, 0, 5) \]
and
\[ \vec{C} = (1, 4, -2) \]

Calculate the following combinations. Express your answers as ordered triplets [e.g., $(9, 4, -2)$].

Part A

**Hint 1. How to add vectors**

Add vectors by adding the $x$ components, $y$ components, and $z$ components individually.

**ANSWER:**

\[ \vec{A} + \vec{B} = (5, -1, 6) \]

Part B

**ANSWER:**

\[ \vec{B} + \vec{C} = (4, 4, 3) \]

Part C

**ANSWER:**

\[ \vec{A} + \vec{B} + \vec{C} = (6, 3, 4) \]
**Part D**

**Hint 1.** Remember the order of precedence

3\(\vec{A}\) means multiply the vector \(\vec{A}\) by the constant 3, which you can do by multiplying each component by 3 separately. Follow normal rules of mathematical precedence; that is, multiply before adding vectors.

**ANSWER:**

\[3\vec{A} + 2\vec{C} = 8, 5, -1\]

**Part E**

**ANSWER:**

\[2\vec{A} + 3\vec{B} + \vec{C} = 14, 2, 15\]

**Part F**

**ANSWER:**

\[2\vec{A} + 3(\vec{B} + \vec{C}) = 16, 10, 11\]

± Vector Addition and Subtraction

In general it is best to conceptualize vectors as arrows in space, and then to make calculations with them using their components. (You must first specify a coordinate system in order to find the components of each arrow.) This problem gives you some practice with the components.

Let vectors \(\vec{A} = (1, 0, -3)\), \(\vec{B} = (-2, 5, 1)\), and \(\vec{C} = (3, 1, 1)\). Calculate the following, and express your answers as ordered triplets of values separated by commas.

**Part A**

**ANSWER:**

\[\vec{A} - \vec{B} = 3, -5, -4\]

**Part B**

**ANSWER:**

\[\vec{B} - \vec{C} = -5, 4, 0\]

**Part C**

**ANSWER:**

\[-\vec{A} + \vec{B} - \vec{C} = -6, 4, 3\]

**Part D**
Part E
ANSWER:

\[ 3\vec{A} - 2\vec{C} = -3, -2, -11 \]

Part F
ANSWER:

\[ -2\vec{A} + 3\vec{B} - \vec{C} = -11, 14, 8 \]

Vector Addition: Geometry and Components

Learning Goal:
To understand how vectors may be added using geometry or by representing them with components.

Fundamentally, vectors are quantities that possess both magnitude and direction. In physics problems, it is best to think of vectors as arrows, and usually it is best to manipulate them using components. In this problem we consider the addition of two vectors using both of these methods. We will emphasize that one method is easier to conceptualize and the other is more suited to manipulations.

Consider adding the vectors \( \vec{A} \) and \( \vec{B} \), which have lengths \( A \) and \( B \), respectively, and where \( \vec{B} \) makes an angle \( \theta \) from the direction of \( \vec{A} \).

In vector notation the sum is represented by

\[ \vec{C} = \vec{A} + \vec{B} \]

Addition using geometry

Part A

Which of the following procedures will add the vectors \( \vec{A} \) and \( \vec{B} \)?

ANSWER:

- Put the tail of \( \vec{B} \) on the arrow of \( \vec{A} \); \( \vec{C} \) goes from the tail of \( \vec{A} \) to the arrow of \( \vec{B} \)
- Put the tail of \( \vec{A} \) on the tail of \( \vec{B} \); \( \vec{C} \) goes from the arrow of \( \vec{B} \) to the arrow of \( \vec{A} \)
- Put the tail of \( \vec{A} \) on the tail of \( \vec{B} \); \( \vec{C} \) goes from the arrow of \( \vec{A} \) to the arrow of \( \vec{B} \)
- Calculate the magnitude as the sum of the lengths and the direction as midway between \( \vec{A} \) and \( \vec{B} \)
It is equally valid to put the tail of $\vec{A}$ on the arrow of $\vec{B}$; then $\vec{C}$ goes from the tail of $\vec{B}$ to the arrow of $\vec{A}$.

**Part B**

Find $C$, the length of $\vec{C}$, the sum of $\vec{A}$ and $\vec{B}$.

Express $C$ in terms of $A$, $B$, and angle $\theta$, using radian measure for known angles.

[Diagram of vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$]

**Hint 1. Law of cosines**

The law of cosines relates the lengths of the sides of any triangle of sides $A$, $B$, and $C$. Using the geometric notation where $c$ is the angle opposite side $C$:

$$C^2 = A^2 + B^2 - 2AB \cos(c).$$

**Hint 2. Interior and exterior angles**

Note that $\theta$ is an exterior angle that is the supplement of angle $C$.

Express $\theta$ in terms of $C$ and relevant constants such as $\pi$, using radian measure for known angles.

ANSWER:

$\theta = \pi - C$

Also accepted: $\sqrt{A^2 + B^2 - 2AB \cos(\pi - \theta)}$

**Part C**

Find the angle $\phi$ that the vector $\vec{C}$ makes with vector $\vec{A}$.

Express $\phi$ in terms of $C$ and any of the quantities given in the problem introduction ($A$, $B$, and/or $\theta$) as well as any necessary constants. Use radian measure for known angles. Use asin for arcsine.

**Hint 1. Law of sines**
Although this angle can be determined by using the law of cosines with $\phi$ as the angle, this results in more complicated algebra. A better way is to use the law of sines, which in this case is

\[
\frac{\sin(\phi)}{B} = \frac{\sin(c)}{C} = \frac{\sin(a)}{A}
\]

ANSWER:

\[
\phi = \arcsin\left(\frac{B\sin(\pi - \theta)}{C}\right)
\]

Also accepted: \(
\arcsin\left(\frac{B\sin(\pi - \theta)}{\sqrt{A^2 + B^2 + 2AB\cos(\theta)}}\right)\) \(\arccos\left(\frac{A + B\cos(\theta)}{C}\right)\) \(\arctan\left(\frac{B\sin(\theta)}{A + B\cos(\theta)}\right)\) \(\arccos\left(\frac{A^2 + C^2 - B^2}{2AC}\right)\)

Addition using vector components

**Part D**

To manipulate these vectors using vector components, we must first choose a coordinate system. In this case choosing means specifying the angle of the $x$ axis. The $y$ axis must be perpendicular to this and by convention is oriented $\pi/2$ radians counterclockwise from the $x$ axis.

Indicate whether the following statement is true or false:

There is only one unique coordinate system in which vector components can be added.

ANSWER:

- $\bigcirc$ true
- $\bigcirc$ false

**Part E**

The key point is that you are completely free to choose any coordinate system you want in which to manipulate the vectors. It is a matter of convenience only, and so you must consider which orientation will simplify finding the components of the given vectors and interpreting the results in that coordinate system to get the required answer. Considering these factors, and knowing that you are going to be required to find the length of $\vec{C}$ and its angle with respect to $\vec{A}$, which of the following orientations simplifies the calculation the most?

ANSWER:

- $\bigcirc$ The $x$ axis should be oriented along $\vec{A}$
- $\bigcirc$ The $x$ axis should be oriented along $\vec{B}$
- $\bigcirc$ The $x$ axis should be oriented along $\vec{C}$

**Part F**

Find the components of $\vec{B}$ in the coordinate system shown.

Express your answer as an ordered pair: $x$ component, $y$ component; in terms of $B$ and $\theta$. Use radian measure for known angles.
Part G

In the same coordinate system, what are the components of $\vec{C}$?

Express your answer as an ordered pair separated by a comma. Give your answer in terms of variables defined in the introduction ($A$, $B$, and $\theta$). Use radian measure for known angles.

ANSWER:

$$C_x, C_y = A + B \cos(\theta), B \sin(\theta)$$

This should show you how easy it is to add vectors using components. Subtraction is similar except that the components must be subtracted rather than added, and this makes it important to know whether you are finding $\vec{C} = \vec{A} - \vec{B}$ or $\vec{C} = \vec{B} - \vec{A}$. (Note that $\vec{B} = -\vec{B}$.)

Although adding vectors using components is clearly the easier path, you probably have no immediate picture in your mind to go along with this procedure. Conversely, you probably think of adding vectors in the way we've drawn the figure for Part B.

This justifies the following maxim: Think about vectors geometrically; add vectors using components.

Vector Components--Review

Learning Goal:
To introduce you to vectors and the use of sine and cosine for a triangle when resolving components.

Vectors are an important part of the language of science, mathematics, and engineering. They are used to discuss multivariable calculus, electrical circuits with oscillating currents, stress and strain in structures and materials, and flows of atmospheres and fluids, and they have many other applications. Resolving a vector into components is a precursor to computing things with or about a vector quantity. Because position, velocity, acceleration, force, momentum, and angular momentum are all vector quantities, resolving vectors into components is the most important skill required in a mechanics course.

The figure shows the components of $\vec{F}_x$, $F_x$ and $F_y$ along the x and y axes of the coordinate system, respectively. The components of a vector depend on the coordinate system's orientation, the key being the angle between the vector and the coordinate axes, often designated $\theta$. 

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Part A

The figure shows the standard way of measuring the angle. \( \theta \) is measured to the vector from the \( x \) axis, and counterclockwise is positive.

Express \( F_x \) and \( F_y \) in terms of the length of the vector \( \vec{F} \) and the angle \( \theta \), with the components separated by a comma.

**ANSWER:**

\[
F_x, F_y = F \cos(\theta), F \sin(\theta)
\]

Also accepted: \( Fx \), \( Fy \), \( \vec{F} \cos(\theta), \vec{F} \sin(\theta), F \cos(\theta), F \sin(\theta) \)

In principle, you can determine the components of any vector with these expressions. If \( \vec{F} \) lies in one of the other quadrants of the plane, \( \theta \) will be an angle larger than 90 degrees (or \( \pi/2 \) in radians) and \( \cos(\theta) \) and \( \sin(\theta) \) will have the appropriate signs and values.

Unfortunately this way of representing \( \vec{F} \), though mathematically correct, leads to equations that must be simplified using trig identities such as

\[
\sin(180^\circ + \phi) = -\sin(\phi)
\]

and

\[
\cos(90^\circ + \phi) = -\cos(\phi)
\]

These must be used to reduce all trig functions present in your equations to either \( \sin(\phi) \) or \( \cos(\phi) \). Unless you perform this followup step flawlessly, you will fail to recognize that

\[
\sin(180^\circ + \phi) + \cos(270^\circ - \phi) = 0.
\]

and your equations will not simplify so that you can progress further toward a solution. Therefore, it is best to express all components in terms of either \( \sin(\phi) \) or \( \cos(\phi) \) with \( \phi \) between 0 and 90 degrees (or 0 and \( \pi/2 \) in radians), and determine the signs of the trig functions by knowing in which quadrant the vector lies.
Part B

When you resolve a vector \( \vec{F} \) into components, the components must have the form \( |\vec{F}| \cos(\theta) \) or \( |\vec{F}| \sin(\theta) \). The signs depend on which quadrant the vector lies in, and there will be one component with \( \sin(\theta) \) and the other with \( \cos(\theta) \).

In real problems the optimal coordinate system is often rotated so that the \( x \) axis is not horizontal. Furthermore, most vectors will not lie in the first quadrant. To assign the sine and cosine correctly for vectors at arbitrary angles, you must figure out which angle is \( \theta \) and then properly reorient the definitional triangle.

As an example, consider the vector \( \vec{N} \) shown in the diagram labeled "tilted axes," where you know the angle \( \theta \) between \( \vec{N} \) and the \( y \) axis.

Which of the various ways of orienting the definitional triangle must be used to resolve \( \vec{N} \) into components in the tilted coordinate system shown? (In the figures, the hypotenuse is orange, the side adjacent to \( \theta \) is red, and the side opposite is yellow.)

![Tilted Axes](image)

(1)  
(2)  
(3)  
(4)

Indicate the number of the figure with the correct orientation.

**Hint 1.** Recommended procedure for resolving a vector into components

First figure out the sines and cosines of \( \theta \), then figure out the signs from the quadrant the vector is in and write in the signs.

**Hint 2.** Finding the trigonometric functions

Sine and cosine are defined according to the following convention, with the key lengths shown in green: The hypotenuse has unit length, the side adjacent to \( \theta \) has length \( \cos(\theta) \), and the side opposite has length \( \sin(\theta) \). The colors are chosen to remind you that the vector sum of the two orthogonal sides is the vector whose magnitude is the hypotenuse; red + yellow = orange.
Part C

Choose the correct procedure for determining the components of a vector in a given coordinate system from this list:

ANSWER:

- Align the adjacent side of a right triangle with the vector and the hypotenuse along a coordinate direction with $\theta$ as the included angle.
- Align the hypotenuse of a right triangle with the vector and an adjacent side along a coordinate direction with $\theta$ as the included angle.
- Align the opposite side of a right triangle with the vector and the hypotenuse along a coordinate direction with $\theta$ as the included angle.
- Align the hypotenuse of a right triangle with the vector and the opposite side along a coordinate direction with $\theta$ as the included angle.

Part D

The space around a coordinate system is conventionally divided into four numbered quadrants depending on the signs of the $x$ and $y$ coordinates. Consider the following conditions:

1. $x > 0$, $y > 0$
2. $x > 0$, $y < 0$
3. $x < 0$, $y > 0$
4. $x < 0$, $y < 0$

Which of these conditions are true in which quadrants?

Write the answer in the following way: If A were true in the third quadrant, B in the second, C in the first, and D in the fourth, enter "3, 2, 1, 4" as your response.
Part E

Now find the components $N_x$ and $N_y$ of $\vec{N}$ in the tilted coordinate system of Part B.

Express your answer in terms of the length of the vector $\vec{N}$ and the angle $\theta$, with the components separated by a comma.

ANSWER:

\[
N_x, N_y = -N \sin(\theta), N \cos(\theta)
\]

Also accepted:

\[-|\vec{N}| \sin(\theta), |\vec{N}| \cos(\theta), -|\vec{N}| \sin(\theta), |\vec{N}| \cos(\theta)\]

± Resolving Vector Components with Trigonometry

Often a vector is specified by a magnitude and a direction; for example, a rope with tension $T$ exerts a force of magnitude $T$ in a direction $35^\circ$ north of east. This is a good way to think of vectors; however, to calculate results with vectors, it is best to select a coordinate system and manipulate the components of the vectors in that coordinate system.

Part A

Find the components of the vector $\vec{A}$ with length $a = 1.00$ and angle $\alpha = 20.0^\circ$ with respect to the $x$ axis as shown.

Enter the $x$ component followed by the $y$ component, separated by a comma.

**Hint 1. What is the $x$ component?**

Look at the figure shown.

$\vec{A}_x$ points in the positive $x$ direction, so $A_x$ is positive. Also, the magnitude $|A_x|$ is just the length $\overline{OL} = \overline{OM} \cos(\alpha)$. 
ANSWER:

\[ \vec{A} = a \cos(\alpha), \ a \sin(\alpha) = 0.940, 0.342 \]

**Part B**

Find the components of the vector \( \vec{B} \) with length \( b = 1.00 \) and angle \( \beta = 20.0^\circ \) with respect to the \( x \) axis as shown.

Enter the \( x \) component followed by the \( y \) component, separated by a comma.

**Hint 1. What is the \( x \) component?**

The \( x \) component is still of the same form, that is, \( L \cos(\beta) \).

ANSWER:

\[ \vec{B} = b \cos(\beta), \ b \sin(\beta) = 0.940, 0.342 \]

The components of \( \vec{B} \) still have the same form, that is, \( (L \cos(\theta), L \sin(\theta)) \), despite \( \vec{B} \)'s placement with respect to the \( y \) axis on the drawing.

**Part C**

Find the components of the vector \( \vec{C} \) with length \( c = 1.00 \) and angle \( \phi = 35.0^\circ \) as shown.

Enter the \( x \) component followed by the \( y \) component, separated by a comma.

**Hint 1. Method 1: Find the angle that \( \vec{C} \) makes with the positive \( x \) axis**

Angle \( \phi = 0.611 \) differs from the other two angles because it is the angle between the vector and the \( y \) axis, unlike the others, which are with respect to the \( x \) axis. What is the angle that \( \vec{C} \) makes with the positive \( x \) axis?

Express your answer numerically in degrees.

ANSWER:
The $x$ component is still of the same form, that is, $L \cos(\theta)$, where $\theta$ is the angle that the vector makes with the positive $x$ axis.

**Hint 2.** Method 2: Use vector addition

Look at the figure shown.

1. $\vec{\xi} = \vec{C}_x + \vec{C}_y$
2. $|\vec{C}_x| = \text{length}(QR) = c \sin(\phi)$
3. $\vec{C}_x$: the $x$ component of $\vec{\xi}$ is negative, since $\vec{C}_x$ points in the negative $x$ direction.

Use this information to find $\vec{C}_x$. Similarly, find $\vec{C}_y$.

**ANSWER:**

$$\vec{\xi} = -c \sin(\phi), \ \cos(\phi) = -0.574, 0.819$$

---

**Finding the Cross Product**

The figure shows two vectors $\vec{T}$ and $\vec{U}$ separated by an angle $\theta_{TU}$.

You are given that $\vec{T} = (3, 1, 0)$, $\vec{U} = (2, 4, 0)$, and $\vec{T} \times \vec{U} = \vec{V}$.

---

**Part A**

Express $\vec{V}$ as an ordered triplet of values, separated by commas.

**ANSWER:**

$$\vec{V} = 0, 0, 10$$

---

**Part B**
Find the magnitude of $\vec{V}$.

**ANSWER:**

$$|\vec{V}| = 10$$

---

**Part C**

Find the sine of the angle between $\vec{J}$ and $\vec{G}$.

**ANSWER:**

$$\sin(\theta_{\vec{J}\vec{G}}) = 0.707$$

---

**± Vector Dot Product**

Let vectors $\vec{A} = (2, 1, -4)$, $\vec{B} = (-3, 0, 1)$, and $\vec{C} = (-1, -1, 2)$.

Calculate the following:

**Part A**

**Hint 1.** Remember the dot product equation

If $\vec{M} = (M_x, M_y, M_z)$ and $\vec{N} = (N_x, N_y, N_z)$, then

$$\vec{M} \cdot \vec{N} = M_x N_x + M_y N_y + M_z N_z$$

**ANSWER:**

$$\vec{A} \cdot \vec{B} = -10$$

**Part B**

What is the angle $\theta_{\vec{A}\vec{B}}$ between $\vec{A}$ and $\vec{B}$?

Express your answer using one significant figure.

**Hint 1.** Remember the definition of dot products

$$\vec{A} \cdot \vec{B} = |\vec{A}| |\vec{B}| \cos(\theta)$$

where $\theta$ is the angle between $\vec{A}$ and $\vec{B}$.

**ANSWER:**

$$\theta_{\vec{A}\vec{B}} = 2 \text{ radians}$$

---

**Part C**

**ANSWER:**
Part D

ANSWER:

\[ 2 \vec{B} \cdot 3 \vec{C} = 30 \]

Part E

Which of the following can be computed?

**Hint 1. Dot product operator**

The dot product operates only on two vectors. The dot product of a vector and a scalar is not defined.

**ANSWER:**

- \( \vec{A} \cdot \vec{B} \cdot \vec{C} \)
- \( \vec{A} \cdot (\vec{B} \cdot \vec{C}) \)
- \( \vec{A} \cdot (\vec{B} + \vec{C}) \)
- \( 3 \cdot \vec{A} \)

\( \vec{V}_1 \) and \( \vec{V}_2 \) are different vectors with lengths \( V_1 \) and \( V_2 \) respectively. Find the following:

Part F

Express your answer in terms of \( V_1 \)

**Hint 1. What is the angle between a vector and itself?**

The angle between a vector and itself is 0.

**Hint 2. Remember the definition of dot products**

\[ \vec{A} \cdot \vec{B} = |\vec{A}| |\vec{B}| \cos(\theta) \text{ where } \theta \text{ is the angle between } \vec{A} \text{ and } \vec{B}. \]

**ANSWER:**

\[ \vec{V}_1 \cdot \vec{V}_1 = V_1 V_1 \]

Also accepted: \( V_1^2 \)

Part G
If \( \vec{V}_1 \) and \( \vec{V}_2 \) are perpendicular,

**Hint 1. What is the angle between perpendicular vectors?**

The angle between vectors that are perpendicular is equal to \( \pi/2 \) radians or 90 degrees.

ANSWER:

\[
\vec{V}_1 \cdot \vec{V}_2 = 0
\]

Also accepted: \( V_1 V_2 \cos \left( \frac{\pi}{2} \right) \)

---

**Part H**

If \( \vec{V}_1 \) and \( \vec{V}_2 \) are parallel,

**Express your answer in terms of \( V_1 \) and \( V_2 \).**

**Hint 1. What is the angle between parallel vectors?**

The angle between vectors that are parallel is equal to 0.

ANSWER:

\[
\vec{V}_1 \cdot \vec{V}_2 = V_1 V_2
\]

---

**± Vector Math Practice**

Let vectors \( \vec{A} = (2, -1, 1) \), \( \vec{B} = (3, 0, 5) \), and \( \vec{C} = (1, 4, -2) \), where \((x, y, z)\) are the components of the vectors along \( \hat{i}, \hat{j}, \) and \( \hat{k} \) respectively. Calculate the following:

**Part A**

**Express your answer as an ordered triplet of components \((x, y, z)\) with commas to separate the components.**

**Hint 1. How to approach this problem**

Components can be multiplied by constants and added up individually.

ANSWER:

\[
2\vec{A} + 3\vec{B} + \vec{C} = 14, 2, 15
\]

**Part B**
Express your answer as an ordered triplet \(|\vec{A}|, |\vec{B}|, |\vec{C}|\) with commas to separate the magnitudes.

**Hint 1. Magnitude of a vector**
Is the magnitude of a vector a scalar quantity or a vector quantity? Recall that a scalar quantity is described simply by a number.

ANSWER:
\(|\vec{A}|, |\vec{B}|, |\vec{C}| = 2.45, 5.83, 4.58

**Part C**

**ANSWER:**
\(\vec{A} \cdot \vec{B} = 11\)

**Part D**

Determine the angle \(\theta\) between \(\vec{B}\) and \(\vec{C}\).

Express your answer numerically in radians, to two significant figures.

**Hint 1. Definition of the dot product**
\(\vec{B} \cdot \vec{C} = B_xC_x + B_yC_y + B_zC_z = |\vec{B}| |\vec{C}| \cos(\theta)\), where \(\theta\) is the angle between vectors \(\vec{B}\) and \(\vec{C}\).

ANSWER:
\(\theta = 1.8\) radians

**Part E**

Express your answer as an ordered triplet of components \((x, y, z)\) with commas to separate the components.

ANSWER:
\(\vec{B} \times \vec{C} = -20, 11, 12\)

**Part F**

ANSWER:
\(\vec{A} \cdot (\vec{B} \times \vec{C}) = -39\)