Chapter 2

Poincare Transformation

We had that the general inertial transformations are linear transformations

\[ x' \mu = \Lambda^\mu_\alpha x^\alpha + a^\mu \] (2.1)

which preserves \( ds^2 \). Let us see what that constraint means. Thus

\[ ds'^2 = ds^2 \] (2.2)

implies

\[ ds'^2 = dx'^\mu \eta_{\mu\nu} dx'^\nu = dx^\mu \eta_{\mu\nu} dx^\nu = ds^2 \] (2.3)

Inserting Eq.2.1 gives

\[ ds'^2 = \Lambda^\nu_\alpha dx^\alpha \eta_{\mu\nu} \Lambda^\mu_\beta dx^\beta = dx^\alpha \eta_{\alpha\beta} dx^\beta \] (2.4)

Equating coefficients of \( dx^\alpha dx^\beta \) gives

\[ \Lambda^\mu_\alpha \eta_{\mu\nu} \Lambda^\nu_\beta = \eta_{\alpha\beta} \] (2.5)

This is the basic constraint on \( \Lambda^\mu_\alpha \) for the transformation to represent an inertial frame of transformation. Thus any linear transformation Eq.2.1 with \( \Lambda^\nu_\mu \) obeying Eq.2.5, and \( a^\mu \)
arbitrary is a transformation between inertial frames and all inertial frame transformations are linear transformations obeying Eq.2.5. These transformations are called

Poincare Transformations \hspace{1cm} (2.6)

and transformations obeying Eq.2.5 with $a^\mu = 0$ are called

Lorentz Transformations \hspace{1cm} (2.7)

(Sometimes Poincare transformations are referred to as Lorentz transformations). It is sometimes convenient to introduce matrix notation. We write for matrix $\Lambda$:

\[(\Lambda)_{\mu\nu} = \Lambda^\mu_{\alpha}; \tilde{\Lambda}^{\alpha\mu} = (\Lambda)_{\mu\alpha} = \Lambda^\mu_{\alpha}\] \hspace{1cm} (2.8)

Eq.2.5 then in the matrix form is:

\[(\tilde{\Lambda})\eta\Lambda = \eta; (\eta)_{\mu\nu} = \eta_{\mu\nu}\] \hspace{1cm} (2.9)

\[det\tilde{\Lambda}det\eta det\Lambda = det\eta\] \hspace{1cm} (2.10)

Hence

\[(det\Lambda)^2 = 1\] \hspace{1cm} (2.11)

or

\[(det\Lambda) = \pm 1\] \hspace{1cm} (2.12)

Hence there are two kinds of Poincare transformation

\[(det\Lambda) = +1, \hspace{0.5cm} \text{“proper transformation”}\] \hspace{1cm} (2.13)

\[(det\Lambda) = -1, \hspace{0.5cm} \text{“improper transformation”}\]
Examples of improper transformation are parity and time reflections.

Parity:

\[ \Lambda^0_0 = 1; \, \Lambda^0_i = 0 = \Lambda^i_0; \, \Lambda^i_j = -\delta^i_j; \]  \hspace{1cm} (2.14)

\[ x'^0 = x^0, \, x'^i = -x^i \]  \hspace{1cm} (2.15)

and one easily checks that Eq.2.14 satisfies Eq.2.5 and so is a Poincare transformation.

Similarly Time reflection:

\[ \Lambda^0_0 = -1; \, \Lambda^0_i = 0 = \Lambda^i_0; \, \Lambda^i_j = \delta^i_j; \]  \hspace{1cm} (2.16)

\[ x'^0 = -x^0, \, x'^i = x^i \]  \hspace{1cm} (2.17)

Transformations with

\[ \Lambda^0_0 > 0, \, \text{“orthochronus transformation”} \]  \hspace{1cm} (2.18)

\[ \Lambda^0_0 < 0, \, \text{“non-orthochronus transformation”} \]

Since \( \det \Lambda \neq 0 \), \( \Lambda \) must have an inverse. Otherwise \( \eta \) has an inverse which we will write as

\[ (\eta^{-1})_{\mu\nu} \equiv \eta^{\mu\nu} = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} = \eta_{\mu\nu} \]  \hspace{1cm} (2.19)

where clearly

\[ \eta^{\mu\alpha}\eta_{\alpha\nu} = \delta^\mu_\nu \text{ or } \eta^{-1}\eta = 1 \]  \hspace{1cm} (2.20)

We can calculate \( \Lambda^{-1} \) using Eq.2.9. Thus multiplying on left by \( \eta^{-1} \) gives

\[ \eta^{-1}\bar{\Lambda}\eta\Lambda = \eta^{-1}\eta = 1 \]  \hspace{1cm} (2.21)
Hence
\[(\Lambda^{-1})_{\mu\alpha} = (\eta^{-1})_{\mu\nu}(\tilde{\Lambda})_{\alpha\beta}(\eta)_{\beta\alpha}\]
(2.22)
or
\[(\Lambda^{-1})_{\mu\alpha} = \eta^{\mu\nu}\Lambda^\beta_{\nu\beta}\eta_{\beta\alpha}\]
(2.23)

Λ⁻¹ is also a Poincare transformation such that
\[x'_{\mu} = (\Lambda^{-1})_{\mu\alpha}x^\alpha + a^\mu\]
(2.24)
obeyes the condition \((\Lambda^{-1})^T\eta(\Lambda^{-1}) = \eta\) (or \((\tilde{\Lambda}^{-1})\eta(\Lambda^{-1}) = \eta\)) for a Poincare transformation. Suppose we had two Poincare transformation:
\[x'^{\mu}_{1} = \Lambda^{\mu}_{1\alpha}x^\alpha + a^\mu_{1}\]
(2.25)
\[x'^{\mu}_{2} = \Lambda^{\mu}_{2\alpha}x^\alpha + a^\mu_{2}\]

with both Λ₁ and Λ₂ obeys \(\tilde{\Lambda}\eta\Lambda = \eta\). If we consider these transformations to be done consecutively:
\[x'^{\mu} = \Lambda^{\mu}_{1\alpha}x^\alpha + a^\mu_{1}\]
(2.26)
\[x'^{\mu}_{2} = \Lambda^{\mu}_{2\alpha}x^\alpha + a^\mu_{2}\]
Then combining those two gives a linear transformation relating \(x'^{\mu}_{2}\) and \(x'^{\mu}\)
\[x'^{\mu}_{2} = \Lambda^{\mu}_{3\alpha}x^\alpha + a^\mu_{3}\]
(2.27)
Λ₃ is called the product of two transformations.