Two-level medium. Summery

\[ \varepsilon_2 - 1 \Rightarrow \]

\[ \omega_0 = \frac{\varepsilon_2 - \varepsilon_1}{\hbar}, \quad H = \hbar_0 + V \]

\[ \varepsilon_1 \rightarrow \frac{1}{12} \]

\[ H_0 = \begin{pmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{pmatrix} \]

\[ V = -\bar{d} \cdot \bar{E} = -\left( \bar{d} \cdot \bar{E}_0 \right) \cdot E \]

\[ \bar{d} \cdot \bar{E}_0 = \begin{pmatrix} 0 & d_{12} \\ d_{21} & 0 \end{pmatrix}, \quad d_{12} \approx 0 < 1 \quad \bar{E}_0 \quad 12 > \]

\[ d_{12} = d_{21}^*; \quad \text{if real}, \quad d_{12} = d_{21} \equiv d. \]

Density matrix equation:

\[ i + \frac{\partial}{\partial t} = \left[ H, \rho \right] = H \rho - \rho H \]

(1) \[ \frac{\partial \rho_{11}}{\partial t} = i \frac{2dE}{\hbar} \left( \rho_{12} - \rho_{21} \right) - \frac{\alpha_{11} - \alpha_{12} \bar{E}}{T} \]

(2) \[ \frac{\partial \rho_{21}}{\partial t} = -i \omega_0 \rho_{21} + \frac{i dE}{\hbar} \cdot \rho_{12} - \frac{\rho_{21}}{T} \]

Add wave equation:

(3) \[ \nabla^2 \bar{E} - \frac{\varepsilon_0 \nabla^2 \bar{E}^2}{c^2} = \frac{4\pi \bar{\rho}}{c^2 \omega^2}, \quad \bar{\rho} = n \cdot Tr \left( \bar{d} \cdot \bar{E}_0 \right) \]

or \[ \bar{\rho} = n d \bar{p}_{12} + n d \bar{p}_{21} \]
Linear response: 1D propagation 11x.

\[ \begin{align*}
E &= \frac{1}{2} E_0 e^{i(kx - \omega t)} + c.c. \quad E_0 (\omega, k) \\
P &= \frac{1}{2} P_0 e^{i(kx - \omega t)} + c.c. \quad P_0 (\omega, k)
\end{align*} \]

\[ P_0 = X(\omega) E_0 \]

From (2):

\[ k^2 - \frac{\omega^2}{c^2} \varepsilon_0 = \frac{\omega^2}{c^2} \rho_0 X(\omega) \]

Assume \( \frac{\partial p}{\partial t} = \text{const} = \Delta p e \) in eqs. (1), (2).

Otherwise, no linear terms are left.

This works when \( E_0 \) is small \( (\ll E_{\text{sat}}) \)

\[ p_{21} = \sigma_{21} e \]

From (2):

\[ \sigma_{21} \left( -i \omega + i \omega_0 + \omega \right) = \frac{i \partial E_0}{2 \hbar} \Delta p e; \quad i \omega = \frac{1}{\tau_2} \]

We will assume \( |\omega - \omega_0| \ll \omega_0 \) close to resonance, \( \Delta p \), and neglect terms \( \sim \epsilon (|\omega + \omega_0| + \omega \omega_0) \) in terms oscillating at negative frequencies \( \omega < 0 \).

\[ \sigma_{21} = \frac{i \partial E_0}{2 \hbar} \Delta p e \left( \frac{1}{\tau_2 + i |\omega_0 - \omega|} \right) \]

Lorentzian shape.
\[ P = nd \left( e^{-i\omega t} + 1 \right) \]

\[ P = \frac{1}{2} P_0 e^{-i\omega t} \]

or,
\[ P_0 = 2nd \delta_{\omega_0} = \chi(\omega) E_0. \]

\[ \chi(\omega) = \frac{i d^2 \text{h} \omega e}{\hbar (\omega_0^2 + \hbar^2 \omega_0 - \omega^2)} \]

\[ \text{h} \omega e = n_{\text{h}}^2 - n_{\text{e}}^2 \equiv n_{\text{e}}^2. \]

When the field is strong, we can consider

monochromatic Fourier components \( E_0, P_0 \).

Moreover, the nonlinear terms \( \sim E_p^2 \) etc.

will inevitably lead to generation of other

frequencies. When we can only about the

effects at \( \omega = \omega_0 \), we can still consider

a quasi-monochromatic wave packet.

\[ E_0(\omega, t) \] has only \( \omega_0 \) component

\[ E_0(x, t) \] is a slowly varying

function of \( x \) and \( t \) as compared to \( E_0 \).
Rotating wave approximation:

\[ E = \frac{1}{2} E_0 e^{i(kx - \omega t)} + c.c. \quad (\text{should be real}) \]
\[ P = \frac{1}{2} P_0 e^{i(kx - \omega t)} + c.c. \quad (\text{should be real}) \]

\[ \mathcal{P}_1 = \mathcal{P}_1 e^{-i\omega t + i\kappa x} \]

Neglect terms with \( e^{-i\omega t} \cdot \frac{\partial^2 E_0}{\partial x^2} \frac{\partial}{\partial t} \)

Then (1) - (3) become:

\[ \frac{\partial \mathcal{E}_1}{\partial t} = - \left( \tau_2 + i(\omega - \omega_0) \right) \mathcal{E}_1 + \frac{i}{\hbar} \frac{\partial E_0}{\partial \mathcal{P}} \frac{\partial \mathcal{P}}{\partial t} \]

\[ \frac{\partial \mathcal{P}}{\partial t} = - \frac{1}{\hbar} \left( \mathcal{P} \frac{\partial}{\partial \mathcal{P}} - \mathcal{P} \frac{\partial}{\partial \mathcal{E}_0} \right) - \frac{i}{\hbar} \text{Im} \left[ E_0 \cdot \mathcal{E}_1 \right] \]

\[ \tau_1 = \frac{1}{\gamma_1}, \quad \tau_2 = \frac{1}{\gamma_2} \]

\[ \frac{\partial E_0}{\partial x} + \frac{\mu}{c} \frac{\partial E_0}{\partial t} = \frac{2 \gamma_1 \omega_0}{\mu e} P_0, \quad \mu = \sqrt{\gamma_0} \]
Some approximate nonlinear solutions

- No relaxation: constant $E_0$; initial response $n_0 = n_0$

\[
\begin{align*}
\frac{\partial n_1}{\partial t} &= -\frac{i\hbar}{\hbar} [E_0^* n_2] \\
\frac{\partial^2 n_1}{\partial t^2} &= -\frac{i\hbar}{\hbar} \text{Im} \left[ E_0^* \frac{\partial n_2}{\partial t} \right] = -\Omega_R^2 n_1
\end{align*}
\]

$\Omega_R^2 = \frac{A^2 |E_0|^2}{\hbar^2}$ - Rabi frequency squared

Rabi oscillations!

Add relaxation:

\[
\begin{align*}
\frac{\partial n_1}{\partial t} &= -\frac{i\hbar}{\hbar} [E_0^* n_2] \\
\frac{\partial^2 n_1}{\partial t^2} &= -\frac{i\hbar}{\hbar} \text{Im} \left[ E_0^* \frac{\partial n_2}{\partial t} \right] = -\Omega_R^2 n_1 - \gamma n_1
\end{align*}
\]
Superradiance:

- no relaxation, \( \omega_0 = \omega \), \( E_0(x,t) \)-real.

Ansatz:

\[
\begin{align*}
\sigma_{21} & = -\frac{i}{2} \sin \varphi \\
\eta_2 & = -\beta \cos \varphi \\
\frac{dE_0}{dt} & = \frac{\partial}{\partial t}
\end{align*}
\]

Then (4) and (5) are identities, and (6) can be transformed to retarded time:

\[
\tilde{t} = t + \frac{\beta}{\omega} x.
\]

Then (6) becomes

\[
\frac{\partial E_0(x,\tilde{t})}{\partial x} = \frac{2 \pi i \omega_0 P_0}{\mu c},
\]

or

\[
\frac{\partial^2 \varphi}{\partial x^2} = \frac{2 \pi \omega_0^2}{\mu c} \sin \varphi: \text{Sine Gordon equation.}
\]

Solution in the form of ultrashort coherent pulses of duration \( T \sim \left( \frac{2 \pi \omega_0 \hbar^2}{\mu c} \right)^{-\frac{1}{2}} \), like an inverted pendulum.
Rate equation approximation:

\[ \frac{\partial \nu_1}{\partial t} = 0 \]

Solve (5) for \( \nu_2 \), substitute into (5), (6),

Also multiply (6) by \( E_0^* \) to get (8).

\[ \frac{\partial h_{11}}{\partial t} = -d_1 \left( h_{11}^2 - h_{12}^2 \right) \quad \frac{d^2 |E_0|^2}{dt^2} \]

\[ \frac{\partial |E_0|^2}{\partial x} + \frac{\partial |E_0|^2}{\partial t} = -\frac{4\pi n_{w} d^2 h_{11}^2}{\mu_{c} c^3} 1 |E_0|^2 \quad \text{(8)} \]

Eq (8) describes field amplification if \( h_{11} < 0 \), i.e.,

\[ h_{11} - h_{12} = h_{11} > 0 \quad \text{population inversion} \]

Mean field approximation: replace \( \frac{\partial |E_0|^2}{\partial x} \) by \( \frac{\partial |E_0|^2}{\partial t} \)

total losses due to absorption in a cavity,

diffractive losses, and losses through mirrors:

\[ \frac{\partial |E_0|^2}{\partial t} + \frac{1}{\tau_c} |E_0|^2 \quad \text{where } \tau_c \text{ is lifetime of photon } \nu_1 \text{ in cavity} \]

Eq (8) becomes

\[ \frac{\partial |E_0|^2}{\partial t} + \frac{1}{\tau_c} |E_0|^2 = q \left| E_0 \right|^2 \quad \text{(9)} \]

where \( |E_0| \) is x-averaged field intensity,

\[ q = \frac{4\pi n_{w} d^2 h_{11}}{\mu_{c} c^3} \quad \text{laser gain in sec}^{-1} \]
Laser threshold: when \( g = \lambda_{\text{tot}} \)

or \( n_{\text{th}} = n_{\text{thr}} = \frac{\lambda_{\text{tot}}}{\lambda_{\text{tot}}} \text{ threshold}
\)

Steady-state laser above threshold:

\[
\frac{\partial n}{\partial t} = 0 \quad \text{from (9) \( \lambda_{\text{tot}} = g \), i.e.,}
\]

\( n_{\text{th}} = n_{\text{thr}} = \text{count above threshold} \)

Note that Eq. (7) can be written as

\[
\frac{\partial n_{\text{th}}}{\partial t} = P - \lambda_{\text{th}} n_{\text{th}} - \frac{d |E_0|^2}{dt} n_{\text{th}}, \quad \text{where}
\]

\( P \equiv \int n_{\text{th}}^2 \text{ is the pumping rate.} \)

In steady state, from (7) or (10) we obtain:

\[
\sqrt{P} = \frac{d |E_0|^2}{dt} = \frac{h_{\text{th}}^2}{h_{\text{th}}} = \frac{h_{\text{th}}^2}{h_{\text{th}}}, \quad \text{or}
\]

\[
|E_0|^2 = |E_{\text{sat}}|^2 \left( \frac{g}{h_{\text{th}}} - 1 \right), \quad \text{or}
\]

\[
|E_{\text{sat}}|^2 = \frac{\lambda_{\text{tot}}}{\lambda_{\text{tot}}} \left( \frac{g}{h_{\text{th}}} - 1 \right), \quad \text{where}
\]

\( |E_{\text{sat}}|^2 = \frac{\lambda_{\text{tot}}}{\lambda_{\text{tot}}} \quad \text{is dimensionless pumping rate.} \)
Note that

\[ \frac{\nu}{\nu_0} = \frac{\nu}{\nu_0} \frac{\nu}{\nu_0} \]

\[ E_0 l \]

\[ l \]

\[ E \]

i.e., the saturation limits the value of \( \nu \) and determines the laser intensity.

\[ \text{Diagram:} \]

- \( E_0 \)
- \( l \)
- \( P \text{thr} \)
- \( P \text{sat} \)
- \( P \)
- \( P \text{thr} \)