• To understand the concept of torque.
• To relate angular acceleration and torque.
• To work and power in rotational motion.
• To understand angular momentum.
• To understand the conservation of angular momentum.
• To study how torques add a new variable to equilibrium.
• To see the vector nature of angular quantities.
What is torque?
When you apply a force to rotate an object about a pivot or an axis, the effectiveness of your action depends not only on the magnitude of the force you apply, but also on a quantity known as the moment arm.

What is the moment arm?
It is the perpendicular distance from the pivot or the axis to the line of action of the applied force.

Torque = (Magnitude of Force) × (Moment Arm)
\[ \tau = Fl \]
Units: N·m
The Magnitude and the Direction of a Torque

Using the examples in the figure on the right:

(a) Torque by $\vec{F}_1$
\[ \tau_1 = F_1 l_1 \]
counter-clockwise, positive

(b) Torque by $\vec{F}_2$
\[ \tau_2 = -F_2 l_2 \]
clockwise, negative

(c) Torque by $\vec{F}_3$
\[ \tau_3 = F_3 l_3 = 0 \]

Torque is a Vector

$\vec{F}_1$ tends to cause *counterclockwise* rotation about point $O$, so its torque is positive: $\tau_1 = +F_1l_1$

$\vec{F}_2$ tends to cause *clockwise* rotation about point $O$, so its torque is negative: $\tau_2 = -F_2l_2$

The line of action of $\vec{F}_3$ passes through point $O$, so the moment arm and hence the torque are zero.
More on the Magnitude of Torque

Three ways to calculate torque:

\[ \tau = Fl = Fr\sin\phi \]

\[ = F(r\sin\phi) = Fr_\perp \]

\[ = (F\sin\phi)r = F_\perp r \]
A rod is pivoted at its center. Three forces of equal magnitude are apply as shown. Which of the following statements correctly describe the order in the magnitudes of the torques by these three forces with respect to the pivot?

(a) $\tau_1 = \tau_2 = \tau_3$  
(b) $\tau_1 > \tau_3 > \tau_2$  
(c) $\tau_2 > \tau_3 > \tau_1$
10.2 Torque and Angular Acceleration

Again, cut the rigid body into many small pieces, A, B, C,... The force acting on piece A is $\vec{F}_A$.

Consider the motion of piece (particle) A. According to Newton’s Second Law,

$$F_{A,tan} = m_A a_{A,tan} = m_A r_A \alpha$$

$$\tau_A = r_A F_{A,tan} = m_A r_A^2 \alpha$$

Sum over the torques for all the pieces:

$$\tau_A + \tau_B + \tau_C \ldots = m_A r_A^2 \alpha + m_B r_B^2 \alpha + m_C r_C^2 \alpha + \ldots = (m_A r_A^2 + m_B r_B^2 + m_C r_C^2 + \ldots) \alpha$$

or

$$\sum \tau = I \alpha$$

This is also known as Newton’s Second Law for rotational motion.
How about torques by internal forces?

\[ \sum \tau_{int} = 0 \]

Therefore, we have

\[ \sum \tau_{ext} = I \alpha \]
Re-visit an Earlier Problem

Example 10.2 on page 288

A cable unwinding from a winch.

Find:
(a) Magnitude of the angular acceleration
(b) Final angular velocity
(c) Final speed of cable

Solution:
The torque by the tension force
\[ \tau = Fl = (9.0 \text{ N}) \times (0.06 \text{ m}) = 0.54 \text{ N} \cdot \text{m}. \]

The moment of inertia about the given axis
\[ I = \frac{1}{2} MR^2 = \frac{1}{2} (50 \text{ kg})(0.06 \text{ m})^2 = 0.090 \text{ kg} \cdot \text{m}^2 \]

Therefore, the angular acceleration
\[ \alpha = \frac{\tau}{I} = (0.54 \text{ N} \cdot \text{m})/(0.090 \text{ kg} \cdot \text{m}^2) = 6.0 \text{ rad/s}^2 \]

Angular Displacement
\[ \theta - \theta_0 = s/r = (2.0 \text{ m})/(0.06 \text{ m}) = 33 \text{ rad} \]

Final angular velocity
\[ \omega = \sqrt{\omega_0^2 + 2\alpha(\theta - \theta_0)} = \sqrt{2\alpha(\theta - \theta_0)} = 20 \text{ rad/s} \]

Final speed of cable
\[ v = r\omega = (0.06 \text{ m})(20 \text{ rad/s}) = 1.2 \text{ m/s} \]
Example 10.3 on page 288  

Given: $M, R, m, \text{ and } h$

Find:  

(a) $\alpha$ for winch, $a$ for bucket, and tension force.  
(b) $v$ for the bucket just before hitting the water  
(c) $\omega$ for the winch just before hitting the water

Solution:

Linear motion of the bucket  

$$mg - T = ma \quad .........(1)$$

Rotational motion of the winch  

$$\tau = I\alpha \quad \text{or} \quad TR = \left(\frac{1}{2}MR^2\right)\alpha \quad .........(2)$$

From (2):  

$$T = \frac{1}{2}MR\alpha = \frac{1}{2}Ma \quad \text{.........................(3)}$$

Substitute (3) into (1):  

$$mg - \frac{1}{2}Ma = ma$$

Linear acceleration:  

$$a = \frac{mg}{m + M/2} = \frac{g}{1 + M/2m}$$

Tension force:  

$$T = Ma/2 = \frac{Mg}{2 + M/m}$$

Angular acceleration:  

$$\alpha = a/R = \frac{g}{R(1 + M/2m)}$$

Final $v$ and $\omega$ calculated using the kinematic equations:

$$v = \sqrt{\frac{2gh}{1 + M/2m}} \quad \text{and} \quad \omega = \frac{1}{R} \sqrt{\frac{2gh}{1 + M/2m}}$$
Example 10.4 on page 290  
Given: Given $M$, $R$, and $h$  
Find: (a) Center of mass acceleration and angular acceleration  
      (b) Tension force  
      (c) Velocity of the center of mass $v_{cm}$  

Solution:  
Apply Newton’s Second Law for linear motion  
$$Mg - T = Ma \quad \text{(1)}$$  

Apply Newton’s Second Law for rotational motion  
$$\tau = I \alpha \quad \text{or} \quad TR = \left(\frac{1}{2}MR^2\right) \alpha \quad \text{(2)}$$  

From (2):  
$$T = \frac{1}{2}MR\alpha = \frac{1}{2}Ma \quad \text{(3)}$$  

Substitute (3) into (1):  
$$Mg - \frac{1}{2}Ma = Ma$$  

Linear acceleration:  
$$a = \frac{2g}{3}$$  

Angular acceleration:  
$$\alpha = a/R = \frac{2g}{3R}$$  

Tension force  
$$T = \frac{Mg}{3}$$  

Final $v_{cm}$ using the kinematic equations  
$$v_{cm} = \sqrt{2ah} = \sqrt{\frac{4gh}{3}}$$
Example 10.5 on page 291: Rolling without slipping
What is the acceleration of a rolling bowling ball?

Given: Given $M$, $R$, and $\beta$
Find: (a) $a_{cm}$ and $\alpha$ (also, $f_s$, required minimum $\mu_s$)

Solution:
Apply Newton’s Second Law for linear motion
$$Mg\sin\beta - f_s = Ma_{cm} \quad \text{(1)}$$
Apply Newton’s Second Law for rotational motion
$$\tau = I\alpha \quad \text{or} \quad f_s R = \left(\frac{2}{5}MR^2\right)\alpha \quad \text{(2)}$$
From (2): $f_s = \frac{2}{5}MR\alpha = \frac{2}{5}Ma_{cm} \quad \text{(3)}$
Substitute (3) into (1):
$$Mg\sin\beta - \frac{2}{5}Ma_{cm} = Ma_{cm}$$
Linear acceleration:
$$a_{cm} = \frac{5}{7}g\sin\beta$$
Angular acceleration:
$$\alpha = \frac{a_{cm}}{R} = \frac{5}{7R}g\sin\beta$$
Static friction force
$$f_s = \frac{2}{5}MR\alpha = \frac{2}{7}Mg\sin\beta$$
Minimum static coefficient of friction
$$\mu_s = \frac{f_s}{n} = \left(\frac{2Mg\sin\beta}{7}\right)/Mg\cos\beta$$
$$= \frac{2}{7}\tan\beta$$
Again, cut the rigid body into many small pieces, A, B, C,… The force acting on piece A is $\vec{F}_A$.

If the angular position of A changes by $\Delta \theta$, the work done by a constant force $\vec{F}_A$ acting on A is:

$$W_A = F_A \tan(R \Delta \theta) = \tau_A \Delta \theta$$

Sum over contributions from all the pieces, the total work

$$W = \tau \Delta \theta = \tau(\theta_2 - \theta_1)$$

A few notes: (a) Applicable for constant torque $\tau$.

(b) Work has units N⋅m.

Power is the rate at which work is done:

$$P = \frac{\Delta W}{\Delta t} = \tau \frac{\Delta \theta}{\Delta t} = \tau \omega$$

(Compare with linear motion in which case $P = Fv$)
10.4 Angular Momentum

In Chapter 8 we defined the momentum of a particle as $\hat{p} = m\hat{v}$, we could state Newton’s Second Law as $\hat{F} = \lim_{\Delta t \to 0} \frac{\Delta \hat{p}}{\Delta t}$.

Here we define the angular momentum of a rigid body:

$$L = I\omega$$

Notes: It is also a vector, same as $\omega$.
Units: kg\cdot m^2/s

Angular momentum of a point particle:

$$L = mvl$$

(kg\cdot m^2/s)

Notes: (a) $l$ is effective the “moment arm”
(b) a particle moving along a straight line can still have an angular momentum about a pivot or rotational axis
Example 10.7 on page 295: A kinetic sculpture

Given: Two small metal sphere of masses $m_1 = m_2 = 2.0$ kg
Uniform metal rod of mass $M = 3.0$ kg and length $s = 4.0$ m
Angular velocity 3.0 rev/minutes about a vertical axis through the middle

Find: Angular momentum and Kinetic energy

Solution:

(a) Angular momentum
Total moment of inertia $I_{\text{total}} = I_{\text{sphere 1}} + I_{\text{sphere 2}} + I_{\text{rod}} = m_1(s/2)^2 + m_2(s/2)^2 + (1/12)Ms^2$

$= 20 \text{ kg}\cdot\text{m}^2$

Angular momentum $L = I\omega = (20 \text{ kg}\cdot\text{m}^2)(3.0\cdot2\pi/60 \text{ s}^{-1}) = 6.2 \text{ kg}\cdot\text{m}^2/\text{s}$

(b) Kinetic energy $K = (1/2)I\omega^2 = 0.96 \text{ J}$
The Relationship Between Torque and Angular Momentum

Since

\[ \sum \tau = I \alpha = I \lim_{\Delta t \to 0} \frac{\Delta \omega}{\Delta t} = \lim_{\Delta t \to 0} \frac{I \Delta \omega}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta (I \omega)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta L}{\Delta t} \]

We can state Newton’s Second Law for rotational motion as

\[ \sum \tau = \lim_{\Delta t \to 0} \frac{\Delta L}{\Delta t} \]

Conservation of Angular Momentum

If \( \sum \tau = 0 \),

Then \( L = \text{constant} \)
Example 10.9 on page 299:

Given:

- Initial body moment of inertia: $I_{\text{body}, i} = 3.0 \text{ kg}\cdot\text{m}^2$
- Final body moment of inertia: $I_{\text{body}, f} = 2.2 \text{ kg}\cdot\text{m}^2$
- Mass of each dumbbell: $m_{\text{dumbbell}} = 5.0 \text{ kg}$
- Initial radius of each dumbbell: $R_{\text{dumbbell}, i} = 1.0 \text{ m}$
- Final radius of each dumbbell: $R_{\text{dumbbell}, f} = 0.20 \text{ m}$
- Initial angular speed: $\omega_i = 1 \text{ rev \ in \ 2 \ s} = 2\pi/2 \text{ rad/s} = \pi \text{ rad/s}$

Find:

(a) $\omega_f$

(b) compare $K_f$ with $K_i$

Solution:

(a) for $\omega_f$

Total initial moment of inertia

$$I_{\text{total},i} = I_{\text{body}, i} + 2I_{\text{dumbbell}, i} = (3.0 \text{ kg}\cdot\text{m}^2) + 2[m_{\text{dumbbell}}(R_{\text{dumbbell}, i})^2] = 13 \text{ kg}\cdot\text{m}^2$$

Total final moment of inertia

$$I_{\text{total},f} = I_{\text{body}, f} + 2I_{\text{dumbbell}, f} = (2.2 \text{ kg}\cdot\text{m}^2) + 2[m_{\text{dumbbell}}(R_{\text{dumbbell}, f})^2] = 2.6 \text{ kg}\cdot\text{m}^2$$

Apply the conservation of angular momentum

$$I_{\text{total},i} \omega_i = I_{\text{total},f} \omega_f$$

so that

$$\omega_f = \frac{I_{\text{total},i} \omega_i}{I_{\text{total},f}} = \frac{5.0\pi \text{ rad/s}}{2.6 \text{ kg}\cdot\text{m}^2} = 2.5 \text{ rev/s}$$

(b) compare $K_f$ with $K_i$

$$K_i = \frac{1}{2}I_{\text{total},i} \omega_i^2 = 64 \text{ J}$$

$$K_f = \frac{1}{2}I_{\text{total},f} \omega_f^2 = 320 \text{ J}$$
The equilibrium of a point particle is determined by the conditions of
\[ \sum F_x = 0 \quad \text{and} \quad \sum F_y = 0. \]

The Equilibrium of a Rigid Body
For the equilibrium of a rigid body, both its linear motion and rotational motion must be considered. Therefore, in addition to
\[ \sum F_x = 0 \quad \text{and} \quad \sum F_y = 0, \]
We must add another condition regarding its rotational motion
\[ \sum \tau = 0. \]
This third condition can be set up about any chosen axis.
Strategy for Solving Rigid Body Equilibrium Problems

General principle:  The net force must be zero.
                 The net torque about any axis must be zero.

- Draw a diagram according to the physical situation.
- Analyze all the forces acting on each part of a rigid body.
- Sketch all the relevant forces acting on the rigid body.
- Based on the force analysis, set up a most convenient $x$-$y$ coordinate system.
- Break each force into components using this coordinates.
- Sum up all the $x$-components of the forces to an equation: $\sum F_x = 0$.
- Sum up all the $y$-components of the forces to an equation: $\sum F_y = 0$.
- Based on the force analysis, set up a most convenient rotational axis.
- Calculate the torque by each force about this rotational axis.
- Sum up all the torques by the forces to an equation: $\sum \tau = 0$.
- Use the above three equations to solve for unknown quantities.
Example 10.12 on page 304
Climbing a ladder (length is 5 m)

Find:
(a) Normal and friction forces at the base of the ladder
(b) Minimum $\mu_s$ at the base
(c) Magnitude and direction of the contact force at the base

Solution: (a) 
\[
\begin{align*}
\sum F_x &= 0 & f_s - n_1 &= 0 \\
\sum F_y &= 0 & n_2 - (800 \text{ N}) - (180 \text{ N}) &= 0 & n_2 = 980 \text{ N} \\
\sum \tau &= 0 & n_1(4.0 \text{ m}) - (180 \text{ N})(1.5 \text{ m}) - (800 \text{ N})(1.0 \text{ m}) &= 0 \\
& & n_1 &= 268 \text{ N} \\
& & f_s &= 268 \text{ N}
\end{align*}
\]

(b) Minimum $\mu_s = (268 \text{ N})/(980 \text{ N}) = 0.27$

(c) 
\[
F_B = \sqrt{(268 \text{ N})^2 + (980 \text{ N})^2} = 1020 \text{ N}
\]
\[
\theta = \tan^{-1} \left( \frac{980 \text{ N}}{268 \text{ N}} \right) = 75^\circ
\]