Mechanical Waves

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Chapter 15
The Wave Equation for \( y(x,t) \)

\[
\frac{\partial^2 y}{\partial x^2} = \mu \frac{T}{\partial t^2} \frac{\partial^2 y}{\partial t^2}
\]

the solutions for \( y(x,t) \) are

\( f(t-x/v) \) and \( g(t + x/v) \)

with \( v = \sqrt{\frac{T}{\mu}} \)
Choosing our favorite solution

\[ y(x, t) = A \cos(kx - \omega t) \]

with

\[ t - \frac{x}{v} = \omega t - kx \]

where

\[ k = \frac{2\pi}{\lambda}, \quad \omega = 2\pi f \quad \text{and} \quad v = \lambda f = \frac{\omega}{k} \]
Traveling waves

\[ A \sin(\omega t - kx) = y(x, t) \text{ for } t = 0 \]

\[ y(x, t) \text{ for } t = t_i > 0 \]

- Amplitude
- Wave length
- Wave moves in +x direction

The period of the wave is the time duration of one cycle.
Particle velocity and acceleration of the medium

\[ y(x, t) = A \cos(kx - \omega t) \]

then

\[ v_y(x, t) = \frac{\partial y(x, t)}{\partial t} = \omega A \sin(kx - \omega t) \]

and

\[ a_y(x, t) = \frac{\partial^2 y(x, t)}{\partial t^2} = -\omega^2 A \cos(kx - \omega t) \]
Energy carried in a wave

\[
F_y = -F \frac{\partial y(x, t)}{\partial x}
\]

with

\[
P(x, t) = F_y(x, t)v_y(x, t) = -F \frac{\partial y(x, t)}{\partial x} \frac{\partial y(x, t)}{\partial t}
\]
Average Power in a sine wave on a string

\[ P(x,t) = \sqrt{\mu F} \, \omega^2 A^2 \sin^2 (kx - \omega t) \]

with

\[ P_{ave} = \frac{1}{2} \sqrt{\mu F} \, \omega^2 A^2 \]

Note: One half of the maximum power transmitted in the wave. Also note the frequency and amplitude dependence of these mechanical waves.
Wave Intensity

Intensity = the time averaged power per unit area

\[ I = \frac{P_{\text{ave}}}{4\pi r^2} \]

units are watts/meter\(^2\)
Reflection from a fixed end and a free end. The phase changes are different for the two reflected waves.

What are the similarities between this effect and what we saw for E&M waves at the interface between regions of different indices of refraction?
Superposition of mechanical waves
The mathematics of wave superposition

\[ y_{total}(x, t) = y_1(x, t) + y_2(x, t) \]

with

\[ y_{total} \] also a solution to the wave equation
Standing Waves on a string

\[ y_1(x,t) = -A \cos(kx + \omega t) \text{ and} \]
\[ y_2(x,t) = A \cos(kx - \omega t) \]

then

\[ y_{\text{total}} = A(-\cos(kx + \omega t) + \cos(kx - \omega t)) \]

using \( \cos(a \pm b) = \cos a \cos b \mp \sin a \sin b \)
Standing waves (continued)

\[ y_{total} = A( - \cos(kx + \omega t) + \cos(kx - \omega t) ) \]

\[ y_{total} = (2A \sin kx) \sin \omega t \]
Location of the nodes for a standing wave on a string.

\[ y_{\text{total}} = (2A \sin kx) \sin \omega t \]

for a string fixed at \( x = 0 \),

the nodes will be located at points where \( kx = m \pi \), with \( x = 0, \lambda / 2, \lambda, 3\lambda / 2, 2\lambda, \ldots \).
Normal Modes of a String

The string is rigidly held at both ends, so we have two sets of boundary conditions to be satisfied.

\[ L = n \frac{\lambda}{2} \]
Normal Modes (continued)

\[ \lambda_n = \frac{2L}{n} \text{ where } n = 1,2,3,4,.. \]

\[ f_1 = \frac{v}{2L} \text{ the fundamental frequency} \]

\[ f_n = n\left(\frac{v}{2L}\right) \text{ give the harmonic frequencies} \]
Standing waves (continued)