Electromagnetic Waves

Chapter 32
November 15, 2012
How do we get the Wave Equation from Maxwell’s Equations???

Consider Maxwell’s Equations when we are far from charges and currents...

These terms will be zero...

The wave equation for EM waves can be derived from Maxwell's equations

\[ \oint E \cdot dA = \text{Gauss' Law} \]
\[ \oint B \cdot dA = 0 \]
\[ \oint B \cdot d\sigma = \mu_0 (\overline{J} + \overline{\epsilon} \frac{\partial \overline{E}}{\partial t}) \text{ (Ampere's Law)} \]
\[ \oint E \cdot d\sigma = -\frac{\partial \overline{B}}{\partial t} \text{ (Gauß's Law)} \]
A plane wave...
The wave equation for EM Radiation

What equations must these time changing $\vec{E}$ vs $\vec{B}$ satisfy?!

Consider a plane wave with

$$\vec{E}(x, t) = E(x, t) \hat{j}$$

$$\vec{B}(x, t) = B(x, t) \hat{k}$$
Now let's apply Faraday's Law to a rectangle in the x-y-plane

\[ \int \mathbf{E} \cdot d\mathbf{l} = E_y(x+\Delta x,t) \Delta y - E_y(x,t) \Delta y \]

\[ = \Delta y \left[ E_y(x+\Delta x,t) - E_y(x,t) \right] \]

\[ = -\frac{d \Phi_m}{dt} \quad \text{(after evaluating this)} \]

\[ \frac{d \Phi_m}{dt} = \frac{d}{dt} \int B \cdot dA = \frac{d}{dt} \int_{x_{min}}^{x_{max}} B_y(x,t) \Delta x \]

\[ = \frac{d}{dt} B_y(x,t) \Delta x \]

\[ \int_{x_{min}}^{x_{max}} \left( \frac{E_y(x+\Delta x,t) - E_y(x,t)}{\Delta x} \right) = \int_{x_{min}}^{x_{max}} \frac{\partial E_y(x,t)}{\partial x} dx = -\frac{d}{dt} B_y(x,t) \]
Now let's look at Ampere's law around a loop in the x-y plane.

\[ \oint B \cdot dl = - \frac{d}{dt} \left[ B_y(x, y, t) \alpha + B_y(x+y, t) \alpha \right] \]

\[ = - \alpha \left[ B_y(x+\Delta x, t) - B_y(x, t) \right] \]

\[ = \mu_0 \epsilon_0 \frac{d}{dt} \phi_E \]

Where

\[ \frac{d\phi_E}{dt} = \frac{d}{dt} \int E \cdot d\hat{A} = \frac{d}{dt} \int E_y(x, t) \alpha \, dx \]

\[ = \mu_0 \epsilon_0 \frac{d\phi_E}{dt} = \mu_0 \epsilon_0 \frac{2E_y(x, t) \alpha \, dx}{dx} \]

\[ = \mu_0 \epsilon_0 \frac{2E_y(x, t) \alpha \, dx}{dx} \]

Again take limit as \( dx \to 0 \)

\[ \frac{dE_y}{dx} = \frac{B_y(x+\Delta x, t) - B_y(x, t)}{\Delta x} \]

\[ \lim_{\Delta x \to 0} \frac{dE_y}{dx} = \mu_0 \epsilon_0 \frac{2E_y(x, t)}{dx} \]

\[ \frac{d}{dx} \left[ B_y(x+\Delta x, t) - B_y(x, t) \right] = \mu_0 \epsilon_0 \frac{2E_y(x, t)}{dx} \]
Using these two results:

\[
\frac{1}{\delta x} \left( \frac{\partial^2 E_y}{\partial x^2} = -\frac{\partial B_y}{\partial t} \right) \Rightarrow \frac{\partial^2 E_y}{\partial x^2} = -\frac{\partial B_y}{\partial t}
\]

\[
\frac{1}{\delta x} \left( -\frac{\partial E_y}{\partial x} = \mu_0 e_0 \frac{\partial^2 E_y}{\partial x^2} \right) \Rightarrow \frac{\partial E_y}{\partial x} = \frac{\mu_0 e_0}{\delta x} \frac{\partial^2 E_y}{\partial x^2}
\]

take the derivative of top with \( x \) and bottom with \( t \).
The Wave Equation for E-fields

Since both have \(-\frac{\partial^2 E_y}{\partial x^2}\) in them we can eliminate two terms to find:

\[ \frac{\partial^2 E_y(x,t)}{\partial x^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E(x,t)}{\partial t^2} \]

with \( \frac{1}{\mu_0 \varepsilon_0} = \omega^2 \) and so our "old friend" the wave equation once again.
So EM waves we find that the waves in general propagate in the direction perpendicular to \( \vec{E} \times \vec{B} \) direction with \( E_y(x,t) = C B_y(x,t) \).
Sinusoidal EM Waves

\[ \mathbf{E}(x,t) = E_{\text{max}} \sin (wt - kx) \hat{\mathbf{j}} \]

\[ \mathbf{B}(x,t) = B_{\text{max}} \sin (wt - kx) \hat{\mathbf{k}} \]

where \( v = \lambda f = \frac{c}{\omega} \)

...as before in mechanical waves...

\[ \mathbf{E}(x,t) = -E_{\text{max}} \sin (wt + kx) \hat{\mathbf{j}} \]

\[ \mathbf{B}(x,t) = B_{\text{max}} \sin (wt + kx) \hat{\mathbf{k}} \]

Remember ... \( \mathbf{E} \times \mathbf{B} \) points in the direction of propagation ...

\[ E_{\text{max}} = c B_{\text{max}} \]
\[ \frac{\partial^2}{\partial x^2} E_y(x, t) = -\frac{2}{\beta} \frac{\partial}{\partial t} B_y(x, t) \]

\[ \frac{\partial}{\partial x} E_x \sin (\omega t - kx) = E_x \frac{\partial}{\partial x} \sin (\omega t - kx) \]

\[ \frac{\partial}{\partial t} B_x \sin (\omega t - kx) = B_x \frac{\partial}{\partial t} \sin (\omega t - kx) \]

\[ E_x(-k) \cos (\omega t - kx) = 0 \]

\[ (\omega + k) B_x \cos (\omega t - kx) = 0 \]

\[ E_x(\omega + k) = B_x \omega \]

\[ E_x = \frac{\omega}{k} B_x \]

\[ E_x = \frac{\omega + k}{\omega} B_x \]

\[ E_x = c B_x \]
Consider….

A sinusoidal EM having a magnetic field of amplitude 1.25 microT and a wavelength of 432 nm is traveling in the +x direction through empty space. (a) What is the frequency of this wave? (b) What is the amplitude of the associated E-field? (c) Write the equations for E and B as a function of x and t.
Energy and Momentum carried by EM waves

Since both E and B fields have stored energy in them, as they move through space they carry energy with them.

Energy density $\rho = \frac{1}{2} \varepsilon_0 E^2 + \frac{1}{2} \mu_0 B^2$ (in vacuum)

and since for EM waves $B = \frac{E}{c} = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} E$

$\rho = \frac{1}{2} \varepsilon_0 E^2 + \frac{1}{2} \mu_0 \left( \frac{E}{c} \right)^2$

$= \frac{1}{2} \varepsilon_0 E^2 + \frac{1}{2} \mu_0 \left( \frac{c}{\sqrt{\mu_0 \varepsilon_0}} E \right)^2$

$= \frac{1}{2} \varepsilon_0 E^2 + \frac{1}{2} \mu_0 (c^2 / (\varepsilon_0 \mu_0)) E^2 = \varepsilon_0 E^2$

$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$
The Poynting Vector

we can now look at the movement (flow) of this energy within the wave

\[ dU = \nabla \cdot \mathbf{E} = \mathbf{A} \cdot \mathbf{E} \, dt \]

\[ = \varepsilon_0 \varepsilon_0 E^2 A \, dt \]

Then

\[ \frac{1}{A} \frac{dU}{dt} = \varepsilon_0 \varepsilon_0 E^2 \]

\[ = \frac{\mathbf{S}}{A} \]

"Poynting Vector"
This new vector \( \vec{S} \) is the energy flow per unit time per unit area and is called the "Poynting Vector"

where:

\[
\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}
\]

and

\[
|\vec{S}| = c \epsilon_0 E^2 = \frac{c \epsilon_0 E^2}{\mu_0} = \sqrt{\frac{\epsilon_0}{\mu_0}} E^2 = \frac{E_0}{\mu_0}
\]

Then the energy flow per unit time (power) is calculated by evaluating the flux of the Poynting vector through the surface of interest

\[
\text{Power} = \oint \vec{S} \cdot d\vec{A}
\]
Time-averaged Poynting Vector or **Intensity**

\[
\bar{S}(x,t) = \frac{1}{\mu_0} \vec{E}(x,t) \times \vec{B}(x,t)
\]

\[
= \frac{E_{\text{max}} B_{\text{max}}}{\mu_0} \cos^2 (kx - \omega t)
\]

So finding the average value as a function of time.

\[
S_{\text{ave}} = \frac{E_{\text{max}} B_{\text{max}}}{2\mu_0} = \text{Intensity}
\]

We define this average value of \( S \) as the "intensity", the averaged power per unit area carried by the wave.
Time-averaged Poynting Vector or \textit{Intensity}

\[ S_{\text{ave}} = \frac{E_{\text{max}} B_{\text{max}}}{2\mu_0} = \frac{E_{\text{max}}^2}{2\mu_0 c} = \frac{1}{2} \varepsilon_0 c E_{\text{max}}^2 \]
Time-averaged Poynting Vector or *Intensity*

- Satellite
- 100 kW transmitter
- $r = 100$ km

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Momentum carried in a wave.

\[ \frac{1}{A} \frac{dp}{dt} = \frac{S}{c} = \frac{E}{\mu_0 c} \]

Flow rate of electromagnetic momentum.
(Per unit area per unit time)

\[ P_{rad} = \frac{S_{ave}}{c} = \frac{E_{max} B_{max}}{2 \mu_0 c} \]

For the average flow rate we get (radiation pressure)
Reflection from surfaces...

for reflection from a shiny surface
\[ \text{Pressure} = \frac{2E}{A} \]

for reflection from a dark surface
\[ \text{Pressure} = \frac{E}{A} \]
Example:

(A) Consider a plane electromagnetic wave with

\[ E_{\text{max}} = 5 \, \text{V/m} \]

Find the corresponding value of \( B_{\text{max}} \) for this wave.

Since \( E_{\text{max}} = c B_{\text{max}} \); \( B_{\text{max}} = \frac{E_{\text{max}}}{c} \)

\[ B_{\text{max}} = \left( 3 \times 10^8 \, \text{m/s} \right) \left( 5 \, \text{V/m} \right) = \]
\[ = 1.66 \times 10^{-8} \, \text{T} \]

(B) What is the intensity of this plane wave?

\[ I = \text{ave} = \frac{E_{\text{max}}}{2 \mu_0} \]

\[ = \left( 5 \, \text{V/m} \right) \left( 1.66 \times 10^{-8} \right) = \]
\[ = 3.31 \times 10^{-8} \, \text{watts/m}^2 \]

(C) You are given that the intensity of light coming from the Sun is 0.75 \( \text{kwatt/m}^2 \). What is the wave \( E \) and \( B \) in this radiation?
\[ I = \text{suns} = 0.78 \times 10^3 \ \text{W/m}^2 = \pm \frac{E_{\text{max}}^2}{mc^2} \]

\[ E_{\text{max}} = 2 \left( 0.78 \times 10^3 \text{W/m}^2 \right) \left( 4 \times 10^7 \text{m} \right) \left( 8 \times 10^9 \right) \]

\[ E_{\text{max}} = 7.66 \times 10^{20} \text{J} \]

\[ B_{\text{max}} = 2.55 \times 10^{-6} \text{m} \]

\( \text{What is the radiation pressure that the sun produces on a black surface/shiny surface?} \)

\[ \text{P_{rad/Black Surface}} = \frac{3\text{suns}}{c} = \frac{0.78 \times 10^3 \text{W/m}^2}{3 \times 10^8 \text{m/s}} \]

\[ = 2.6 \times 10^{-6} \text{ Pascal} (\text{N/m}^2) \]

\[ \text{P_{rad/Shiny surface}} = \frac{2 \times 3\text{suns}}{c} = 5.2 \times 10^{-6} \text{ Pascal} (\text{N/m}^2) \]