Chapter 21
Electric Charge and
Electric Field
September 4, 2012
Recitations and labs start this week. There has been a correction to the lab # that you will be doing. Static Electricity is lab 1 not lab 2.

Your labs will meet every other week for two hours instead of weekly for one hour.

Chapter 21 homework due date shifted to Thursday morning, 9/6.
Notice the similarities of the Law of Universal Gravity and Coulomb’s Law for the electric force.

\[ \vec{F}_{\text{coulomb}} = k \frac{q_1 q_2}{r^2} \hat{r} \]

\[ \vec{F}_{\text{gravitation}} = G \frac{M_1 M_2}{r^2} \hat{r} \]
Introduction to E-fields…

- We have seen the idea of a force field before when studying the law of gravity.
  - The gravitational force field surrounded a massive object producing a gravitational attraction on any other object with mass.
  - Gravitational Field, $g = \frac{F_G}{M_{\text{test object}}}$
The Electric Field

A diagram showing a charge $Q$ emitting electric field lines, with points $P$, $a$, $b$, $c$, $+Q$, $F_a$, $F_b$, and $F_c$. The arrows indicate the direction of the electric field at those points.
E-fields (continued)

- We can ask the same sort of question of the Coulomb force on a test charge produced by another charge, Q.
  - \( \text{E-field} = \frac{F_{\text{coulomb}}}{q_{\text{test}}} = \frac{k Q}{r^2} \), for a point charge.
- The E-field can be a function of position.
- E-field is a vector quantity!
  - Points away from + charges
  - Points toward - charges
E-fields (continued)

- If we somehow know the E-field produced by a charge distribution, then the force that this distribution has on a point charge, q, will be \( F = q E \).

- Units of Electric Field are Newtons/Coulomb.
Electric Field for a point charge

\[ \vec{E}_{\text{point charge}} = k \frac{q}{r^2} \hat{r} \]
Reminder about Unit Vectors

\[ \hat{r} = \cos \theta \hat{i} + \sin \theta \hat{j} \]

\[ \hat{\theta} = - \sin \theta \hat{i} + \cos \theta \hat{j} \]
Example 1

Suppose you have a point charge of 3.0 μCoulombs in a UNIFORM E-field of 3.5 N/C along the x-axis. What is the force that this field exerts on the point charge?
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Suppose you have a point charge of 3.0 microCoulomb in a UNIFORM E-field of 3.5 N/C along the x-axis. What is the force that this field exerts on the point charge?

\[ F = qE \]

\[ = (3.0 \text{ muC})(3.5 \text{ N/C}) \]

\[ = 10.5 \times 10^{-6} \text{ N in the + x direction} \]
Example 2

- Now consider a point charge located at the origin. If the charge is $-3.5 \times 10^{-6}$ C, find the electric field value at the coordinates $(0.3 \text{ m}, 0.4 \text{ m})$ in the xy-plane.
Solution

\[ E_x = -k \frac{Q}{(x^2 + y^2)} \cos(\theta) \]

\[ = 9 \times 10^9 \{- 3.5 \times 10^{-6}/.5^2\}(.3/.5) \]

\[ = -7.56 \times 10^4 \text{ N/C} \]

\[ E_y = -k \frac{Q}{(x^2 + y^2)} \sin(\theta) \]

\[ = -1.00 \times 10^5 \text{ N/C} \]
Let's now consider the Electric Field that is produced by a collection of point charges, $q_1, q_2, q_3, q_4$.

$$d_n = \sqrt{(x-x_n)^2 + (y-y_n)^2 + (z-z_n)^2}$$

"Distance formula in 3D"
Returning to our discussion of Electric Fields.

Let’s look at what happens when you have a collection of point charges together. What is the total E-field?

\[ \vec{F}_{Q} = Q \cdot \vec{E} \text{ due to charge} \]

Distance between \( Q \) and point where you wish to find the E-field.

Distance vector: \( \vec{d} = \vec{r}_{2} - \vec{r}_{1} \)

\[ d_{1} = (x-x_{1})\hat{i} + (y-y_{1})\hat{j} + (z-z_{1})\hat{k} \]

The magnitude of the distance:

\[ |\vec{d}| = \sqrt{(x-x_{1})^2 + (y-y_{1})^2 + (z-z_{1})^2} \]

A unit vector along this direction:

\[ \hat{d}_{1} = \frac{\vec{r}_{2} - \vec{r}_{1}}{|\vec{d}|} = \text{unit vector pointing away from } Q. \]
So now we need to evaluate the individual Coulomb forces between all points of particle with the test charge.

\[ \vec{F}_{\text{from } q_1} = \vec{F}_{q_2} + \vec{F}_{q_3} + \vec{F}_{q_4} \]

\[ = \frac{kq_1q_1}{r_1^2} \hat{r}_1 + \frac{kq_2q_2}{r_2^2} \hat{r}_2 + \frac{kq_3q_3}{r_3^2} \hat{r}_3 + \frac{kq_4q_4}{r_4^2} \hat{r}_4 \]

The electric field is then found from our definition:

\[ \lim_{q \to 0} \frac{\vec{F}}{q} = \vec{E}_{\text{total (due to } q \text{ charges)}} \]

\[ \lim_{q \to 0} \frac{\vec{F}_1}{q_1} = \frac{kq_1q_1}{r_1^2} \hat{r}_1 + \frac{kq_2q_2}{r_2^2} \hat{r}_2 + \frac{kq_3q_3}{r_3^2} \hat{r}_3 + \frac{kq_4q_4}{r_4^2} \hat{r}_4 \]

This is just the sum of the individual E-field due to the 4 individual qk charges added together.

\[ \vec{E}_{\text{total}} = \vec{E}_{q_1} + \vec{E}_{q_2} + \vec{E}_{q_3} + \vec{E}_{q_4} \]
How can we treat a continuous charge distribution and find its E-field? 

Continuous Charge Distribution Example:

- Line charge distribution
- Surface charge distribution
- Volume charge distribution

\[ \lambda = \text{charge/length} \]

\[ \sigma = \text{charge/area} \]

\[ \rho = \text{charge/vol} \]
- How much charge is contained on a segment of wire charge 5 cm long if it has a charge/unit length \( \lambda = 2 \times 10^3 \text{ coul/m} \)?

\[ q = 1.0 \times 10^{-10} \text{ C} \]

- How much charge is contained on a square plate of side 0.1 \( \times \) 0.1 m if it has a surface charge density \( \sigma = 3 \times 10^6 \text{ coul/m}^2 \)?

\[ q = 3.0 \times 10^{-8} \text{ C} \]

- How much charge is contained in a cylinder 0.5 m long and 0.1 m in diameter if in this volume there is a charge density \( \rho = 2 \times 10^{-3} \text{ coul/m}^3 \)?

\[ q = 7.85 \times 10^{-6} \text{ C} \]
For point charge we had
\[ \vec{E}_{TOTAL} = \sum \frac{Q_i}{\varepsilon_0 \vec{d}_i} \]

For continuous charge we get
\[ \vec{E}_{TOTAL} = \int \frac{\kappa Q_{\text{d}r}}{4\pi \varepsilon_0 (r - r')} dr' \]

Charge density

Point where you want to find field

Distance from element to field location

Unit vector pointing from charge element to field location
Let's try an example and see how to put all this together.

Charge element $dq' = \lambda \, dx'$

**Step 1**

$\ dq' = \lambda \, dx'$

**Step 2**

$d = x - x'$

**Step 3**

Unit vector from $dq'$

$\hat{d} = \frac{(x - x')\hat{i}}{|(x - x')^2|}$
Then
\[ E_{\text{TOTAL}}^{(x)} = \int_{-a}^{+a} \frac{k x \ dx'}{(x-x')^2} \]

This can be integrated directly by making the substitution \( u = (x-x') \) if \(-du = dx'\)

\[ \int -\frac{ku}{u^2} \ dx' = \frac{kx}{u} \]

So
\[ E_{\text{TOTAL}} = \frac{Kx}{(x-a)} \ i \bigg|_{-a}^{+a} = \frac{Kx}{(x-a)} - \frac{Kx}{(x+a)} \]
Let's try another example.

Find the E-field for our line charge segment any where in the xy plane.

\[ dq' = \lambda \, dx' \]

\[ d = \sqrt{(x-x')^2 + (y)^2} \]

\[ \vec{d} = \frac{(x-x')\hat{i} + y\hat{j}}{\sqrt{(x-x')^2 + y^2}} \]

**STEP 1**

**STEP 2**

**STEP 3**
\[ E_{\text{total}}(x,y) = \int_{-a}^{+a} \frac{K d x'}{(\sqrt{(x-x')^2+y^2})^2} \left( \frac{(x-x')^2+y^2}{(x-x')^2+y^2} \right) \]

**NOTE THE INTEGRAL IS REALLY 2 INTEGRALS**

An \( x \) component & a \( y \) component.

\[ E_x \text{ total} (x,y) = \int_{-a}^{+a} \frac{K d x'}{(x-x')^2+y^2} \]

\[ E_y \text{ total} (x,y) = \int_{-a}^{+a} \frac{K d x'}{(x-x')^2+y^2} \]
From the integral table at the back of the book.

\[ \int \frac{dx}{(x^2 + a^2)^{3/2}} = \frac{1}{a^2} \frac{x}{(x^2 + a^2)^{1/2}} \]

These are the two we need:

\[ \int \frac{x \, dx}{(x^2 + a^2)^{3/2}} = -\frac{1}{(x^2 + a^2)^{1/2}} \]

for y component.

for x component.
The X component

Using these integrals we can now complete our calculations for $E_x$.

$$E_x(x, y) = \int_{-\infty}^{+\infty} \frac{K \lambda (x-x') dx'}{(x-x')^2+y^2)^{\frac{3}{2}}} = -K\lambda \left[ \frac{-1}{((x-k)^2+y^2)^{\frac{3}{2}}} \right]_{-\infty}^{+\infty}$$

$$= K\lambda \left[ \frac{1}{(x-a)^2+y^2} - \frac{1}{(x+a)^2+y^2} \right]$$
The Y component

\[ E_y(x, y) = \int_{-\infty}^{+\infty} \frac{k_1 y \, dx'}{(x-x')^2 + y^2 k_2} = \frac{x}{y} \frac{k_1 (x-x')}{(x-x')^2 + y^2 k_2} \]

\[ = -\frac{k_1}{y} \int_{-\infty}^{+\infty} \left[ \frac{(x-a)}{(x-a)^2 + y^2 k_2} - \frac{(x+a)}{(x+a)^2 + y^2 k_2} \right] dx' \]
How do you use this technique in other charge geometries?

For surface:

\[ \mathbf{E}(p) = \oint_{\text{surface}} \mathbf{E} \cdot d\mathbf{a} \]

For volume:

\[ \mathbf{E}(p) = \int_{\text{volume}} \mathbf{E} \cdot d\mathbf{v} \]
Let's go through the process of calculating the Electric Field due to a continuous distribution of charge.

**EXAMPLE**

Find the E-field due to a ring of charge with total charge $+Q$ and radius $R$ at any point on the axis passing through the center of the ring and perpendicular to the plane of the ring.

![Diagram of a ring charge with electric field lines](image)

Step 1) E-field for a point charge

$$\vec{E}(P) = \int \frac{kQ \vec{e}}{r^2}$$
Step 2) Divide the continuous charge distribution into small point charge elements, $dq'$. 

Step 3) Find value of an infinitesimal point charge in terms of parameters given. 
$$dq' = \lambda dl = \left( \frac{\epsilon_0}{4\pi} \right) \rho dl$$

Step 4) Calculate distance from point charge to where we wish to find the $E$-field. 
$$distance = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$$

Step 5) Find the vector direction of the charge $dq'$ at the point where you want to find the field.

Our convention... 

"$E$" points away from $(x,y)$ a positive charge.

the vector pointing away from $dq'$ to $(x,y)$ is given by the following vector subtraction. 

$$\vec{d} = \vec{r} - \vec{r}' = (x-x')\hat{i} + (y-y')\hat{j} + (z-z')\hat{k}$$
Step 1) Substitute quantities into the following integral for the total E-field.

\[ \vec{E}(p) = \int \frac{d\vec{E}}{\text{point charge}} = \int \frac{K dq}{(r - r')^2} \]

For the problem at hand

\[ dq' = \lambda \, dl' = \lambda \, R \, d\theta' \quad \text{with} \quad \lambda = \frac{\sigma}{2\pi R} \]

(distance) \[ l = \sqrt{(x-x')^2 + (y-y')^2 + z^2} \]

(direction) \[ \frac{r - r'}{l} = \frac{(x-x')^2 + (y-y')^2 + z^2}{(x-x'^2 + (y-y')^2 + z^2)} \]

with \( x' = R \cos \theta' \) and \( y' = R \sin \theta' \)

and the integral over \( \theta' \) from 0 to \( 2\pi \).
So we get the following:

\[ E(x) = \int_0^{2\pi} \frac{k\lambda R \delta'(-R \cos \theta)}{(R^2 + z^2)^{3/2}} \left( R^2 \cos^2 \theta + R^2 \sin^2 \theta + \frac{z^2}{R^2} \right) \]

\[ E_x = \int_0^{2\pi} \frac{k\lambda R \delta'(-R \cos \theta)}{(R^2 + z^2)^{3/2}} \]

\[ E_y = \int_0^{2\pi} \frac{k\lambda R \delta'(-R \sin \theta)}{(R^2 + z^2)^{3/2}} \]

\[ E_z = \int_0^{2\pi} \frac{k\lambda R \delta'(-R \cos \theta)}{(R^2 + z^2)^{3/2}} \frac{z}{R^2} \]

\[ E_z = \frac{k\lambda R \frac{z}{R^2}}{(R^2 + z^2)^{3/2}} \]
**Electric Field Lines**

A graphical way of indicating the strength and direction of the electric field due to a charge distribution.

a) The electric field direction in a location in space is always tangent to the E-field line at that point.

b) The strength of the electric field in space is proportional to the number of E-field lines per unit area in that spot.

c) Field lines start on positive charges and end on negative.

d) Field lines never cross.
E-field for a negative charge

Region of uniform E-field

Region of non-uniform E-field

Field gets weaker in this direction

Field gets stronger this way
Field lines start on + charges and end on – charges…...
Field lines cannot cross...
Density of lines is proportional to the strength of the field....
Polarization of a dipole in an electric field demo

http://www.youtube.com/watch?v=1jP_D0S2CtY
Dipoles

- **Dipole moment**
  - $p = q d$
  - (pointing from the negative charge to the positive charge)

- **Torque on a dipole**
  - $\tau = p \times E = (pE \sin \phi)$
  - $U = -p \cdot E = -(pE \cos \phi)$