Off-shell Supersymmetry
and the
M-theory Effective Action

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with
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Open string effective action

- Tseytlin’s attempt at open superstring gives a weird answer.
- Cecotti and Ferrara (1986): 4D, N=1. They found a \( \mathcal{L}_{CF} = \mathcal{Z}_D^2 \Psi W^2 + \mathcal{Z}_D^4 \Psi \bar{W}^2 \) that reproduced Tseytlin’s result but also another: Born-Infeld again!
- Metsaev, Rahkmanov, and Tseytin realize the mistake (R boundary condition) and recalculate with NS to find that the low-energy effective action is Born-Infeld (1987).
- Bagger and Galperin demonstrate that, starting in 4D, N=1 superspace, a second non-linearly-realized supersymmetry uniquely fixes the action to be Born-Infeld (1996).

\[ \mathcal{L}_{CF} = \int d^2 \theta W^2 + \int d^4 \theta \Psi \bar{W}^2 \bar{W}^2 \]

\[ F^2 + F^4 + 8 \text{ supercharges} \Rightarrow \text{Born-Infeld} \]
Closed string effective action?

- Can we take analogous steps for closed strings/gravity?
- Cannot do Tseytlin et al. calculation: World-sheet theory is interacting (quartic) even for constant Riemann tensor.
- Unclear what the gravitational analog of Born-Infeld is. Constant field strength? No birefringence? No ghosts? $\sqrt{\text{determinant form}}$? Higher dimensions? Supersymmetric? ...
- Analog of BG result may hold: There are only two supersymmetric densities in flat space. They would be related by $N \geq 2$ supersymmetry. Maybe even fixed by $N$ large enough?
M-theory effective action?

- 11D SG has maximal D and N.
- First higher-derivative correction is of order $(Riemann)^4$.
- Higher-derivative correction requires a higher-derivative correction to the supersymmetry transformation, which requires higher derivatives in the effective action, ... → ∞.
- It is believed that in this case the analog of the Born-Infeld statement holds:

$$R + R^4 + 11D, \quad N = 1 \text{ supersymmetry } \Rightarrow \text{Leading M-theory LEEA?}$$

Supersymmetrization order-by-order in components is unfeasible. Off-shell supersymmetry might fix this, but supersymmetry is on shell if $\#Q > 9$ (Berkovits 1993).

- Perhaps keeping some $0 < N < 1$ manifest in 11D is enough?
Previous work on fractional superspaces


- The vector multiplet in this 10D, $N=1/4$ superspace consists of a real field $V(x, \theta, \bar{\theta}; y)$ familiar from 4D, $N=1$ and an SU(3) triplet of chiral superfields $\Phi_i(x, \theta; y)$.

- Action $\sim \int d^4 x d^6 y \int d^2 \theta W^\alpha W_\alpha +$ Chern-Simons terms.


- Seiberg and Witten (1994): Starting with 4D, $N=2$ SYM in terms of $N=1$ superfields, there is a D-term integral of the Kähler function $K(\Phi, \bar{\Phi})$ and an F-term integral of the gauge kinetic term determined by a holomorphic gauge kinetic function $f(\Phi)$.

- The second non-manifest supersymmetry determines the former in terms of the latter: $K = -i/2 f'\bar{\Phi} + h.c.$ Imposing $N=4$ fixes $f(\Phi)$ uniquely.

- 5D, $N=1/2$ linearized supergravity was constructed with Luty and Phillips (2002) and used to compute loop corrections to slepton masses in a phenomenological analog of heterotic M-theory (with also Buchbinder, Gates, Goh, and Ng).

- Extended to 11D SG with group at Texas A&M.

- **Goal:** Higher-derivative corrections to the M-theory effective action.
11D, N=1/8 Superspace

We construct 11D supergravity in an off-shell superspace \( M \sim X \times Y \) with \( X \) a 4D, N=1 supermanifold and \( Y \) a bosonic 7D manifold.

Decompose 11D supergravity fields

\[
\begin{align*}
    e_m^a & \rightarrow e_m^a, A_m^i, g_{ij} \\
    \psi_m^\alpha & \rightarrow \psi_m^\alpha, \psi_m^{\alpha i}, \psi_j^\alpha, \psi_j^{\alpha i}, \\
    C_{mnp} & \rightarrow C_{mnp}, C_{mni}, C_{mij}, C_{ijk}
\end{align*}
\]

with \( a, m = 0, 1, 2, 3 \)

and \( i, j = 1, \ldots, 7 \)

and \( \alpha = 1, 2 \)
Step 2 (of 6) : Reinterpreting the Metric on $Y$

- Rewrite the metric $g_{ij}$ on $Y$ in terms of a 3-form (Bryant 1987, Hitchin, Joyce 2000)

$$\sqrt{gg_{ij}} = \frac{1}{144} \varepsilon^{k_1 \ldots k_7} \varphi_{i k_1 k_2} \varphi_{j k_3 k_4} \varphi_{k_5 k_6 k_7}$$

- This 3-form is “stable” when $g_{ij}$ is invertible. A stable 3-form on a 7 manifold is a(n almost) $G_2$ structure.

- The torsion of this $G_2$ structure is defined by $\nabla_i \varphi_{jk} = T^m_{ij}(\varphi)_{km}$.

- If $T^j_i = 0$, the holonomy of $\nabla$ is contained in $G_2$. The $49 = 1 + 7 + 14 + 27$ components of this tensor make up the torsion forms

$$d\varphi = \tau_0 \ast \varphi + 3\tau_1 \land \varphi + \ast \tau_3 \quad \text{and} \quad d(\ast \varphi) = 4\tau_1 \land \ast \varphi + \tau_2 \land \varphi$$

If, furthermore, $Y$ is compact and simply-connected, then $\text{Hol}(\nabla) = G_2$.

- The functional $\int_Y \sqrt{g(\varphi)} = \int_Y \varphi \land \ast \varphi$ is called the Hitchin (2000) functional. When $d\varphi = 0$ its equation of motion $d(\ast \varphi) = 0 \Leftrightarrow G_2$ holonomy metrics ("Topological M-theory" Dijkgraaf, Gukov, Neitzke, Vafa 2004).
Step 3 (of 6): Superfield Embedding

Embed the bosonic fields into 11D, N=1/8 superfields:

- $e_m^a$ and $\psi_m^a$ in a real 4-vector-valued superfield $U^a(x, \theta, \bar{\theta}; y)$ familiar from (conformal) 4D, N=1 supergravity,
- $\psi_{m i}^{a i}$ into a spinor superfield $\Psi_{a i}(x, \theta, \bar{\theta}; y)$ in the $\tilde{7}$ of GL(7),
- $A_m^i$ into a real superfield $V^i(x, \theta, \bar{\theta}; y)$: a vector on $\mathcal{Y}$,
- $C_{mnp}$ into a real scalar superfield $X(x, \theta, \bar{\theta}; y)$: a 0-form on $\mathcal{Y}$,
- $C_{mni}$ into 7 chiral spinors $\Sigma_{ai}(x, \theta; y)$: a 1-form on $\mathcal{Y}$,
- $C_{mi j}$ into 21 real superfields $V_{ij}(x, \theta, \bar{\theta}; y)$: a 2-form on $\mathcal{Y}$,
- $C_{ijk} + i\phi_{ijk}$ into 35 chiral superfields $\Phi_{ijk}(x, \theta; y)$: a 3-form.

All superconformal primary fields of the 4D, N=1 superconformal algebra.
Step 4 (of 6): Chern-Simons Action

The last five form a “non-abelian tensor hierarchy” of superfields. Such things have a unique Chern-Simons-like invariant supersymmetrizing the term \( \int C \wedge dC \wedge dC \):

\[
-12\kappa^2 L_{CS} = i \int d^2 \theta E \left[ i\Phi_{[0,3]} \wedge G_{[4,4]} + \Sigma^\alpha_{[2,1]} \wedge W_{[2,6]\alpha} \right] \\
+ \int d^4 \theta E \left[ V_{[1,2]} \wedge H_{[3,5]} - X_{[3,0]} F_{[1,7]} \right] + \text{h.c.}
\]

- \( \Phi \wedge G = G\Phi \wedge d\Phi + \cdots \) contains “\( G_2 \) superpotential” (Gukov 1999).
- \( \Phi \wedge G = \frac{1}{2} \Phi \wedge W^\alpha \wedge W_\alpha + \cdots \) also contains the holomorphic gauge-kinetic function \( f(\Phi) = \Phi \) familiar from Seiberg-Witten theory.

These terms and the others (10 total) are required for gauge invariance.

Unique invariant containing F-term integrals:

A gauge-invariant superpotential or gauge-kinetic function.
There is a natural lift of the Hitchin functional to 11D, N=1/8 superspace:

$$S_H = -\frac{3}{\kappa^2} \int d^7 y \int d^4 x \int d^4 \theta E(\bar{G}G)^{1/3} \sqrt{g(F)} f(x) , \quad x = \frac{H_i g^{ij}(F) H_j}{(\bar{G}G)^{2/3}}$$

Uniquely fixed by local supersymmetry on X, diffeomorphism invariance on Y, tensor hierarchy gauge invariance, engineering dimensions, conformal weights, \( U(1)_R \) weights, and GL(7) representation.

Introducing the gravitino superfield, we can extend the local supersymmetry on X to all of M. This fixes \( f(x) \) exactly:

The combination \( \hat{f} := f - 2xf'' \) satisfies the quartic equation

$$\frac{x}{4} \hat{f}^4 + \hat{f}^3 - 1 = 0$$

and

\( f(0) = 1 \Rightarrow f(x) \).
Checks

We have subjected this result to the following consistency checks:

- Linearizing around a flat background, projecting to components, and throwing out fermions, we recover the correct bosonic component action of eleven-dimensional supergravity.

- Taking $M = \mathbb{R}^{4|4} \times Y$ for general $Y$, reducing to components, throwing out all fermions and those bosons with polarizations in 4D directions, we obtain the “scalar potential”

  \[ V = \frac{1}{4} d_X^2 + 2g_{ij} d^i d^j - d_{7ij}^2 + \frac{1}{2} d_{14ij}^2 - \frac{1}{18} |f_{1ijk}|^2 + \frac{i}{2} d_k F_{ijk}^2 - \frac{1}{24} |f_{7ijk}|^2 + \frac{1}{24} |f_{27ijk}|^2 \]

  D- and F-term supersymmetry conditions $\Leftrightarrow G_2$-holonomy and zero flux.

- Substituting the on-shell values of the auxiliary fields,

  \[ V(x, y) = -\frac{21}{16} \tau_0^2 - 15 \tau_1^2 + \frac{1}{8} \tau_2^2 + \frac{1}{24} \tau_3^2 + \frac{7}{4} \sigma_0^2 + 9 \sigma_1^2 + \frac{1}{24} \sigma_3^2 \]

  $G_2$ identities for torsion forms and C-field analogs reduces this to

  \[ V(x, y) = -\frac{1}{2} R(g) + \frac{1}{4 \cdot 4!} F_{ijkl}^2 \]

  $=\text{Einstein-Hilbert and Maxwell terms on } Y \text{ from auxiliary field mechanism.}$
Can we guess higher-derivative terms? Many are easy. Most interesting one:

- We want to covariantize the term $\int C \wedge X_8$.
- It contains $\int C \wedge R \wedge R \wedge R$.
- Decomposing, it contains $\int C \wedge W_X \wedge W_X \wedge W_Y \wedge W_Y$ (Weyl tensors of $X, Y$).
- $W_X \Leftrightarrow$ gravitino curl $W_{\alpha \beta \gamma}$ from 4D, N=1 supergravity.
- So term $\Leftrightarrow \varepsilon^{ijklmn} \int \Phi_{ijk} W_{\alpha \beta \gamma} W_{\alpha \beta \gamma} S_{lmnp}$ for some $S \sim (W_Y)^2$.

But $S$ must be chiral, and the Riemann tensor on $Y$ is real. This was also the case for the potential, so perhaps it is generated by some auxiliary field mechanism?

But the Weyl tensor is in the $\{70^\prime\}$ representation of $G_2$, and there is no auxiliary field for this!
Strategies

- We are pursuing a few strategies to study this problem.
  - Construct Born-Infeld in 10D, N=1/4 superspace.
  - Auxiliary fields in SU(3) reps:
    - \{3\} \Leftrightarrow \text{chiral } \Phi_i \Leftrightarrow (2,0) \text{ part } F_{ij} \text{ of the curvature of the connection in a Calabi-Yau compactification.}
    - a singlet \Leftrightarrow \text{vector field } V \Leftrightarrow \text{trace of (1,1) part } F_k^k.
  - F-term SUSY \Leftrightarrow \text{holomorphic connex.}
    D-term SUSY \Leftrightarrow \text{Donaldson-Uhlenbeck-Yau equation.}
  - The analog of the \{70'\} above is the \{8\} = \binom{6}{2} - 3 - 3 - 1 \leftrightarrow F_i^j - \frac{1}{3} \delta_i^j F_k^k
    (the rest).
- Solution to the \{8\}-problem in BI \Leftrightarrow \text{solution to } \{70'\}-problem in M-theory?
  Better yet: If we can show there is no solution to the \{8\}-problem, we can forget about the \{70'\}-problem and go work on something else.
Shut up and calculate

1. Construct off-shell supergeometry.
2. Construct $R \wedge R$ super-4-form.
3. Square to get $R \wedge R \wedge R \wedge R$.

To see how this works, first do easier 5D, $N=1/2$ superspace:

2. We then use the known supergeometry (with Gates and Phillips 2003) to construct $R \wedge R$ (Hanaki, Ohashi, and Tachikawa 2006):

\[
G = W^{\alpha\beta\gamma} W_{\alpha\beta\gamma}
\]
\[
\mathbb{H}_i = -\frac{1}{4} D^\gamma F'_i{}^{\alpha\beta} W_{\alpha\beta\gamma} + \frac{1}{8} F'_i{}^{\alpha\beta} \bar{D}^{\bar{\alpha}} D_{\beta} G_{\bar{a}} + \frac{1}{8} \bar{D}^{\bar{\alpha}} F'_i{}^{\alpha\beta} D_{\beta} G_{\bar{a}}
\]
\[
+ \frac{1}{16} \left( D^\beta \bar{D}_{\bar{\alpha}} F'_i{}^{\alpha\beta i} + \frac{1}{4} D_{\alpha} \bar{D}^2 \bar{\lambda}_{\bar{\alpha} i} \right) G_{\alpha}^{\bar{a}} + \frac{1}{32} D^2 \bar{\lambda}_{\bar{\alpha} i} D_{\alpha} R + \text{h.c.}
\]

3. This step is trivial in 5D.

Conveniently, this answer can be lifted to 11D, $N=1/8$ unchanged.
Summary

- We have embedded eleven-dimensional supergravity in an off-shell 11D, N=1/8 superspace.
- We have constructed the linearized off-shell supergeometry for it by starting with the fields and their gauge and local supersymmetry transformations and constructing the Riemann tensor, gravitino curls, and super-4-form field strength.
- We have partially constructed the \((\text{Riemann})^2\) invariants with the intention of using them to construct the leading \(\sim R^4\) M-theory corrections.
- Along the way, we have revisited the 5D, N=1/2 analog of this problem and are currently studying the 10D Born-Infeld analog.
Outlook

- Success in the construction of even the most non-trivial part of the $\int C \wedge X_8$ term is not the end of the story:
  - Chern-Simons-like action is not invariant under all 11D super-diffeomorphisms (cf. 2-derivative action).
  - D-term integral that fixes this problem already determined by $f(x)$. The non-invariance patched up by shift $G \rightarrow G - \frac{1}{4} \text{tr} R \wedge R$ of tensor hierarchy field strengths (Witten 1996).
  - But D-term integrand is non-polynomial $\Rightarrow$ non-polynomial dependence in Riemann.

- Outcome expected from general arguments:

  Supersymmetrization of $\mathbb{R}^4$ must be non-polynomial.
Outlook

- This would be an 11D supergravity analog of Born-Infeld. If this is correct, it would settle many questions about the definition of the latter but also leave some open and suggest new ones:
  - It is not constructed purely from the curvature. Some derivatives thereof will be generated by the superspace integration. These are needed for supersymmetry.
  - There are no ghosts.
  - What is the fate of the singular GR solutions? This should not be difficult to answer once the action is known.
  - Can the bosonic part be put in \( \sqrt{\text{determinant}} \) form? Does it define a new characteristic class?
  - Is it the leading low-energy effective action of M-theory? This could be checked by comparing terms available in the literature from e.g. amplitude computations.
  - Is there a 64-supercharge theory for which the M-theory result is the Goldstone? Witten's 12D theory and E\(_8\) framing (1996)?

- ... 

- In addition, there are many applications/extensions (superstrings, compactifications, SFT, ...).
THANK YOU!
\[ \delta_0 \Psi_{\alpha i} = \Xi_{\alpha i} + \bar{G} G_{ij} \nabla_\alpha \Omega^j + i W_{\alpha ij} \bar{\Omega}^j, \]
\[ \Delta_1 \Phi = \Xi^\alpha \wedge W_\alpha + \tilde{\nabla}^2 \left( \frac{i}{2} \iota_\Omega U + \frac{1}{4} \iota_\bar{\Omega} (F \wedge H) \right), \]
\[ \Delta_1 V = -\frac{i}{2} \bar{G} \iota_\Omega F + \frac{i}{2} G \iota_\Omega F, \]
\[ \Delta_1 \Sigma_\alpha = -G \Xi_\alpha, \]
\[ \Delta_1 X = -\frac{i}{2} \bar{G} \iota_\Omega H + \frac{i}{2} G \iota_\Omega H, \]
\[ \delta_1 \nu = -\frac{1}{2} \bar{G} \Omega - \frac{1}{2} G \bar{\Omega}. \]

\[ \mathcal{U}_{ijkl} := 3 \epsilon_{ijklmnp} \frac{\partial}{\partial F_{mnp}} \left( \sqrt{g} f (G \bar{G})^{-1/3} \right) \]
\[ G_{ij} = (G \bar{G})^{-1/3} \left( \hat{f}^{-1} g_{ij} + \frac{i}{2} (G \bar{G})^{-1/3} \hat{F}_{ijkl} g^{kl} H_l \right) \]

\[ \delta_0 \Psi_{\alpha i} = \Xi_{\alpha i} + g_{ij} D_\alpha \Omega^j, \]
\[ \delta_1 \Phi_{ijk} = \frac{1}{2i} \tilde{\varphi}_{ijk} \bar{D}^2 \bar{\Omega}^l, \]
\[ \delta_1 V_{ij} = \frac{1}{2i} \varphi_{ijk} (\Omega^k - \bar{\Omega}^k), \]
\[ \delta_1 \Sigma_{\alpha i} = -\Xi_{\alpha i}, \]
\[ \delta_1 \nu^i = -\frac{1}{2} (\Omega^i + \bar{\Omega}^i). \]
NATH field strengths and Bianchi identities

\[ E = \partial \Phi \]
\[ F = \frac{1}{2i} (\Phi - \Phi) - \partial V \]
\[ W_\alpha = -\frac{1}{4} \tilde{\nabla}^2 \nabla_\alpha V + \partial \Sigma_\alpha + \iota_{\mathcal{W}_\alpha} \Phi \]
\[ H = \frac{1}{2i} \left( \nabla^\alpha \Sigma_\alpha - \tilde{\nabla}_\alpha \cdot \tilde{\Sigma}^{\alpha'} \right) - \partial X - \omega_h(\mathcal{W}, V) \]
\[ G = -\frac{1}{4} \tilde{\nabla}^2 X + \iota_{\mathcal{W}_\alpha} \Sigma_\alpha \]

\[ 0 = -\partial E \]
\[ \frac{1}{2i} (E - \bar{E}) = \partial F \]
\[ -\frac{1}{4} \tilde{\nabla}^2 \nabla_\alpha F = -\partial W_\alpha - \iota_{\mathcal{W}_\alpha} E \]
\[ \frac{1}{2i} \left( \nabla^\alpha W_\alpha - \tilde{\nabla}_\alpha \cdot \tilde{W}^{\alpha'} \right) = \partial H + \omega_h(\mathcal{W}, F) \]
\[ -\frac{1}{4} \tilde{\nabla}^2 H = -\partial G - \iota_{\mathcal{W}_\alpha} W_\alpha \]
\[ \tilde{\nabla}_\alpha G = 0 \]

\[ \omega_h(\chi_\alpha, v) := \iota_{\chi^\alpha} \nabla_\alpha v + \iota_{\tilde{\chi}^{\alpha'}} \tilde{\nabla}^{\alpha'} v + \frac{1}{2} \left( \iota_{\nabla^\alpha \chi_\alpha} v + \iota_{\tilde{\nabla}^{\alpha'} \tilde{\chi}^{\alpha'}} v \right) \]
Chern-Simons-like invariant

\[-12L_{CS} = i \int d^2 \theta \mathcal{E} \Phi [EG + \frac{1}{2} W^\alpha W_\alpha - \frac{i}{4} \tilde{\nabla}^2 (FH)]
\]
\[+ \int d^4 \theta EV \left[ (E + \bar{E}) H + \omega(W, F) - i \nabla^\alpha F \omega_{\alpha} F + i \tilde{\nabla}_\alpha \cdot F \omega_{\alpha} \cdot F \right]
\]
\[+ i \int d^2 \theta \mathcal{E} \Sigma^\alpha \left[ EW_\alpha - \frac{i}{4} \tilde{\nabla}^2 (F \nabla_\alpha F) \right]
\]
\[\int d^4 \theta EX \left[ (E + \bar{E}) F \right] + \text{h.c.} \]