Properties of superfluids.

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1 Normal and superfluid density

Let the gas of excitations moves with the velocity $\mathbf{v}$with respect to the liquid as a whole (drift velocity). What momentum per unit volume $\mathbf{P}$ does it carry? The same value can be identified as the mass current $\mathbf{j}$. By definition it is expressed in terms of the distribution finction of excitations $f(\mathbf{p},\mathbf{v})$ (the occupation number of an excitation with fixed momentumb$\mathbf{p}$ and fixed drift velocity$\mathbf{v}$:

$$\mathbf{j} \equiv \mathbf{P} = \int f(\mathbf{p},\mathbf{v})\,\mathbf{p}\frac{d^3p}{(2\pi\hbar)^3} \tag{1}$$

In the state of equilibrium with drift velocity equal to zero the distribution function has the Bose-Einstein form:

$$f(\mathbf{p},0) = f_{BE}(\varepsilon) = \frac{1}{e^{\frac{\varepsilon}{T}}-1} \tag{2}$$

where, in the case of weakly interacting Bose-gas $\varepsilon = \varepsilon(p) = \sqrt{s^2p^2 + \left(\frac{p^2}{2m}\right)^2}$ is the energy of excitation, $s = \sqrt{gn/m}$ is the sounnd velocity and T is the temperature. Generally the excitation energy is not so simple, but the equation (2) is still valid. In the case of zero drift velocity the current (1) is equal to zero since the energy is symmetric with respect to the change $\mathbf{p} \to -\mathbf{p}$. When the drift velocity is not zero, the energy of excitations that are well determined in the frame of the liquid as a whole are subject to the Galilean transformation (see either your records or the file on my website called "Weakly interacting Bose-gas"):

$$\varepsilon(\mathbf{p},\mathbf{v}) = \varepsilon(\mathbf{p}) - \mathbf{pv} \tag{3}$$

If the drift velocity is small enough, it is possible to expand the occupation number over small value $\mathbf{pv}$ restricting the expansion by the linear terms:

$$f(\mathbf{p},\mathbf{v}) = f_{BE}(\varepsilon - \mathbf{pv}) = f_{BE}(\varepsilon) - \mathbf{pv}\frac{df_{BE}}{d\varepsilon} \tag{4}$$

Further I omit the subscript "BE" for brevity. Substituting the occupation numbers given by eq. (4) into integral for the current eq. (1), we see that the first term vanishes and the current is a linear function of the drift velocity. For a component j_a we find:

$$j_\alpha = -v_\beta \int p_\alpha p_\beta \frac{df}{d\varepsilon} \frac{d^3p}{(2\pi\hbar)^3} \tag{5}$$

Since the liquid is isotropic, it is invariant under change of sign of any component of momentum separately from others. Therefore the integral in eq. (5) does not vanish only for diagonal components with $\alpha = \beta$. Each such a component is equal to $1/3$ of the integral in which square of one momrntum component, for example p_x^2 is replaced by the square of vector $\mathbf{p}^2$. Let denote this value by ρ_n:

$$\rho_n = -\frac{1}{3} \int \mathbf{p}^2 \frac{df}{d\varepsilon} \frac{d^3p}{(2\pi\hbar)^3} \tag{6}$$

Then the mass current or momentum per unit volume transferred by excitations reads $\mathbf{j} = \rho_n \mathbf{v}$. This equation shows that the value ρ_n has a meaning of the mass per unit volume associated with the excitations. Since the excitations can participate in reactions of decay and merging, they form a normal system in which dissipation is possible. Thus, ρ_n can be called normal density. The mass current associated with the excitations is normal current and the drift velocity of the gas of excitations in the Lab frame of references will be denoted $\mathbf{v}_n$. In the general case when the liquid as a whole moves with the velocity $\mathbf{v}_s$ the normal mass current or the momentum per unit volume with respect to the liquid as a whole reads:

$$\mathbf{j}_n = \rho_n \left(\mathbf{v_n} - \mathbf{v}_s\right) \tag{7}$$

The total mass current reads:

$$\mathbf{j} = \rho_\mathbf{n} \left(\mathbf{v_n} - \mathbf{v_s}\right) + \rho \mathbf{v_s} \tag{8}$$

Introducing the notation for superfluid density:

$$\rho_s = \rho - \rho_n \tag{9}$$

we transform the eqution for the total current to the most convenient and physically transparent form:

$$\mathbf{j} = \rho_n \mathbf{v}_n + \rho_s \mathbf{v}_s \tag{10}$$

This relationship is the base of the so-called two-fluid model of the superfluid liquid. The liquid consists of two inter-penetrable fluids, normal and super. They contribute independently to the total mass density and the total mass current. One immediate consequence of the system of equations (6-10) is that the superfluid density at $T = 0$ coincides with the total density. Indeed, at zero temperature there is no excitations. Therefore $\rho_n = 0$ and form eq. (9) we find $\rho_s = \rho$. At the transition point just opposite $\rho_s = 0$ and $\rho_n = \rho$.

Hydrodynamic equations of motion consist of two conservation laws: continuity equation for the total density (conservation of the total number of particles) and conservation of the total momentum that I will not discuss here. Additionally they used the fact of the coherence of the condensate. It means that the condensate can be described by its macroscopic wave function $\Psi(\mathbf{r}, t)$. The superfluid velocity is associated with the gradient of phase ϕ of the wave function $\left(\Psi = |\Psi| e^{i\phi}\right)$:

$$\mathbf{v}_s = \frac{\hbar}{m} \nabla \phi \tag{11}$$

Two-fluid model was first proposed by L. Tissa in 1938. The hydrodynamics of the two-fluid liquid was developed by Landau in 1941. One of important consequences is the possibility of wave-like propagation of temperature (second sound) due to superflow. The superflow convection is responsible for the absence of boiling below the superfluid transition temperature. Observation of the second sound as a temperature oscillation was proposed by E. Lifshitz and experimentally realized by Peshkov in 1946.

2 Quantization of vortices in a superfluid.

Everybody observed vortices in normal liquid. In the superfluid they have special features. In vortices the normal component experiences the friction (viscosity) and stops, but super-component can rotate without dissipation in statioary regime. Eq. (11} shows that superfluid velocity is gradient of some potential. The continuity equation

$$\frac{\partial \varrho}{\partial t} + \nabla \mathbf{j} = 0 \tag{12}$$

in stationary regime requires that $\nabla \mathbf{j} = 0$. The superfluid density in a vortex does not change. Therefore the phase ϕ satisfies the equation $\nabla^2 \phi = 0$. A simplest cylindrically symmetric solution of this equation with singularity at the axis of the vortex line $r = 0$ is $\phi = n\varphi$ where φ is the asymuthal angle and n is a constant factor. For the superfluid it cannot be arbitrary since the wave function of the condensate is proportional to $e^{i\phi}$ and it must be a single-valued function of coordinates. It happens only if n is an arbitrary non-zero integer. Then the velocity defined by eq. (11) reads:

$$\mathbf{v}_s = \frac{\hbar n \hat{\varphi}}{mr} \tag{13}$$

Note that the circulation of the velocity is quantized:

$$\gamma = \oint \mathbf{v}_s d\mathbf{r} = \frac{2\pi \hbar n}{m} \tag{14}$$

Let calcualte the energy of such a vortex per unit length of the vortex line. It is given by the integration of kinetic energy:

$$E_v = \pi \int_a^L \rho_s v_s^2 r dr = \frac{\pi \rho_s \hbar^2 n^2}{m^2} \ln \frac{L}{a} \tag{15}$$

Here the upper limit of integration L is the transverse size of vessel; the lower limit a is a microsopic size, for example distance between particles. This result shows that, at a fixed circulation, the vortex with $|n| > 1$ is unstable. It decays into $|n|$ vortices with $|n| = 1$. At such process the energy decreases since instead factor n^2 in eq. (15) a factor $|n|$ appears. Thus, further we consider only two values of $n = \pm 1$ to which we refer as vortices and anti-vorices. Quantization of the vortex circulation in a superfluid was first established by L. Onsager in 1947.

3 Vortex lattices in a rotating superfluid

Feynman in 1955 have considered the problem of a rotating superfluid. First his conclusion was that a superfluid in a rotating vessel whose walls interact with the liquid should rotate in such a way that the average angular velocity of liquid is equal to the angular velocity of the vessel. In a vessel rotating with the angular velocity $\mathbf{\Omega}$ the condition of thermodynamic equilibrium is that the thermodynamic potential $F_{rot} = F - \mathbf{M\Omega}$ has minimum. In this respect it is similar to the situation at fixed pressure instead of fixed volume. Then the Gibbs thermodynamic potential Φ instead of Helmholtz free eenrgy F has minimum. F is the potential in variables T, V and $\mathbf{M}$. Its differential includes the term $\mathbf{\Omega} d\mathbf{M}$ and it takes a minima in the variable $\mathbf{\Omega}$ at a fixed $\mathbf{M}$. In contrast the potential F_{rot} is a potential in the variable $\mathbf{\Omega}$ so that its differential includes the term $-\mathbf{M} d\mathbf{\Omega}$. Differentiating the potential F_{rot} over $\mathbf{M}$, we find the equilibrium condition:

$$\frac{\partial F}{\partial \mathbf{M}} = \mathbf{\Omega} \tag{16}$$

The derivative $\frac{\partial F}{\partial \mathbf{M}}$ is the average angular velocity of the superfluid. Eq. (16) proves the Feynman statement.

How this average rotation may be realized in a superfluid with its zero viscosity? Feynman has proposed that it can be realized as a regular 2-dimensional lattice of the quantized vortex lines. The angular velocity can be defined as the circulation of the velocity per unit area which follows from the relation $\mathbf{\Omega} = \frac{1}{2} \nabla \times \mathbf{v}$ and the Stokes theorem applied to a plane surface of unit area perpendicular to the direction of the vector $\mathbf{\Omega}$. Thus, to have the same circulation, it is necessary to locate $n = \Omega m / 2\pi\hbar$ vortices per unit area. This relation defines the area of elementary cell and the lattice constant.

The vortex lattice was observed in the rotating superfluid Helium. The vortex lines approacing to the surface of the superfluid Helium on the boundary with the equilibrium He gas change locally the shape of the surface due to interplay between gravitational, centrifugal and surface tension forces. There appear deeps at the end of each vortex line. These deeps were observed optically with microscope and by observation of the Laue diffraction from 2d lattice.