The saddle point (steepest descent) method.

This is a method of asymptotic calculation working for integrals of the type:

$$I(\Lambda) = \int_{\Gamma} e^{\Lambda f(z)} g(z) \, dz$$  \hspace{1cm} (1)

where $\Lambda$ is a large positive number; $f(z)$ and $g(z)$ are analytic functions of the complex variable $z$ and $\Gamma$ is a contour in the complex plane $z$. The idea of the saddle point method is based on the fact that analytic functions do not have maxima and minima of their moduli in the range of their analyticity, they have only saddle points. In the saddle point the function has a maximum when going along the direction of the steepest decent (see Figure). If the contour $\Gamma$ can be deformed in such a way that it passes through the saddle point in the direction of the steepest descent, the large value of parameter $\Lambda$ ensures that the essential contribution to the integral comes from a small vicinity of the saddle point since the exponent quickly decreases and the detailed shape of the curve of the steepest decent far from the saddle point is not important. In the saddle point the derivative $f'(z_0) = 0$. The direction of the steepest descent is determined by the requirement that the difference $\text{Re} f(z) - \text{Re} f(z_0)$ is negative. With precision of the first non-trivial quadratic term it reads as follows:

$$f''(z_0)(z - z_0)^2 < 0$$  \hspace{1cm} (2)

The argument of a complex number $Z$ is determined by its polar representation: $Z = |Z| e^{i \arg Z}$. The argument of the expression in equation (2) must be $\pm \pi$ to ensure that it is negative. It implies that:

$$\arg (z - z_0) = \pm \frac{\pi}{2} - \frac{1}{2} \arg f''(z_0)$$  \hspace{1cm} (3)
This equation determines the direction of steepest descent. Assuming that the contour $\Gamma$ can be deformed properly, and expanding the function $f(z)$ over the powers of $z - z_0$ to the quadratic term, we find the approximate value of the integral $I(\Lambda)$:

$$I(\Lambda) \simeq 2g(z_0) e^{\Lambda f(z_0)} \int_\gamma \exp \left[ -\frac{\Lambda}{2} |f''(z_0)| |z - z_0|^2 \right] dz \quad (4)$$

where $\gamma$ is a ray at the angle $\alpha = \frac{\pi}{2} - \frac{1}{2} \arg f''(z_0)$ to the real axis from the saddle point; the function $g(z)$ changes slowly and, therefore, is taken at the saddle point. Let introduce a new integration variable $\zeta = |z - z_0|$. Then $dz = d\zeta e^{i\alpha}$. The integral in (4) is reduced to the Gaussian integral and can be calculated explicitly:

$$I(\Lambda) \simeq \pm 2g(z_0) e^{\Lambda f(z_0)} e^{i\alpha} \sqrt{\frac{\pi}{2\Lambda |f''(z_0)|}} = \pm ig(z_0) e^{\Lambda f(z_0)} \sqrt{\frac{\pi}{2\Lambda f''(z_0)}} \quad (5)$$

This is the final result of our calculation. Note that the range of $|z - z_0|$ substantially contributing to the integral are confined by the constraint $\frac{1}{2} |f''(z_0)| |z - z_0|^2 \lesssim 1$ since at larger $|z - z_0|$ the exponent quickly decreases. Assuming that $|f''(z_0)| \sim 1$, we find for essential range of integration $|z - z_0| \sim 1/\sqrt{\Lambda}$. To analyze what are corrections to this result, let us consider the next (cubic) term of the expansion of the expression in the exponent $e^{\Lambda f(z)}$. It is $\frac{1}{2} f'''(z_0) (z - z_0)^3$. In the essential range of integration it has the order of magnitude $1/\sqrt{\Lambda} \ll 1$ if $|f'''(z_0)| \sim 1$. This is the magnitude of the relative value of correction to the main contribution (5).