Partial waves in scattering theory. Faxen-Holzmark formula.

1. Partial wave amplitudes

The scattering amplitude $f(\theta)$ for a spherically symmetric scatterer depends only on the polar angle θ and does not depend on the azymuthal angle φ. Therefore it can be expanded into a Fourier-like series over Legendre polynomials:

$$f(\theta) = \sum_{l=0}^{\infty} (2l+1) f_l P_l(\cos\theta) \tag{1}$$

The coefficients f_l are called partial amplitudes. The different spherical harmonics (angular moments l) contributes independently in the scattering amplitude. It is incorrect for the differential cross section $\frac{d\sigma}{d\Omega} = |f(\theta)|^2$. Indeed, this value depends on all possible products $f_l f_{l'}^*$. Nevertheless, the partial waves contribute independently into the total cross section $\sigma = \int |f(\theta)|^2 \sin\theta d\theta$. Due to ortogonality of the Legendre polynomials, the total cross-section reads:

$$\sigma = 4\pi \sum (2l+1) |f_l|^2 \tag{2}$$

Thus, it is reasonable to introduce partial cross section:

$$\sigma_l = 4\pi (2l+1) |f_l|^2 \tag{3}$$

The calculation of the partial amplitudes is an important part of the partial waves scattering theory. It is reduced to the calculation of the phase shifts of radial wave functions.

2. Solution of Schrödinger equation with spherically symmetric potential and its asymptotic form

The solution of the stationary SE for the scattering problem in the case of spherically symmetric potential reads:

$$\psi(r,\theta) = \sum_{l=0}^{\infty} C_l R_l(r) P_l(\cos\theta) \tag{4}$$

The radial wave functions $R_l(r)$ obey the radial SE:

$$\frac{d^2 R_l}{dr^2} + \frac{2}{r}\frac{dR_l}{dr} + \left(k^2 - \frac{l(l+1)}{r^2} - v(r)\right) R_l = 0 \tag{5}$$

Asymptotically at $r \to \infty$ they approach the free spherical wave solutions:

$$R_l(r) \cong \frac{\sin\left(kr - \frac{l\pi}{2} + \delta_l\right)}{r} \tag{6}$$

The constant δ_l is the phase shift of the partial $l-$th wave due to the potential. It is equal to zero for a free particle. The coefficients C_l can be found from the requirement that asymptotically at $r \to \infty$, the wave function (4) approaches the sum of incident and spherical outgoing wave:

$$\psi(r,\theta) \cong e^{ikz} + f(\theta)\frac{e^{ikr}}{r} \tag{7}$$

Important! The ingoing spherical wave is only due to the incident wave e^{ikz}. Comparing the coefficients at $P_l(\cos\theta)$ and ingoing spherical wave for e^{ikz} and $\psi(r,\theta)$, one can find the coefficients C_l. Thus, we arrive at the problem of asymptotical behavior of the plane wave at $r \to \infty$. The solution of this problem (see Appendix) is:

$$e^{ikz} \cong \frac{e^{ikr}}{2ikr}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta) - \frac{e^{-ikr}}{2ikr}\sum_{l=0}^{\infty}(2l+1)(-1)^l P_l(\cos\theta) \tag{8}$$

The spherical waves $\frac{e^{\pm ikr}}{ikr}$ are multiplied by factors $\frac{1}{2}\sum_{l=0}^{\infty}(2l+1)(\pm1)^l P_l(\cos\theta) = \delta(1 \mp \cos\theta)$ that have quite resonable physical meaning: the plane wave visible from the origin looks like an ingoing wave moving from the left $(\theta = \pi)$ and as an outgoing wave moving to the right $(\theta = 0)$. Now let us compare eq. (8) with the aymptotic of the function $\psi(r,\theta)$given by eqs. (4,6):

$$\psi(r,\theta) \cong \frac{e^{ikr}}{2ir}\sum_{l=0}^{\infty}C_l e^{-i\frac{l\pi}{2}+i\delta_l}P_l(\cos\theta) - \frac{e^{-ikr}}{2ir}\sum_{l=0}^{\infty}C_l e^{i\frac{l\pi}{2}-i\delta_l}P_l(\cos\theta) \tag{9}$$

The ingoing wave in this asymptotic must be the same as the ingoing wave of the insident plane wave, i.e. the second terms in eqs. (8,9) must coincide. Therefore, the partial waves cocoefficients must coinside. That gives:

$$C_l = \frac{e^{i\frac{l\pi}{2}+i\delta_l}}{k}(2l+1) \tag{10}$$

Plugging them into the term of the outgoing wave, one finds:

$$\psi(r,\theta) \cong \frac{e^{ikr}}{2ikr}\sum_{l=0}^{\infty}(2l+1)e^{2i\delta_l}P_l(\cos\theta) - \frac{e^{-ikr}}{2ikr}\sum_{l=0}^{\infty}(2l+1)(-1)^l P_l(\cos\theta) \tag{11}$$

Comparing this formula with eq. (7) and employing the asymptotic of the plane wave (8), one arrives at the following equation for the scattering amplitude:

$$f(\theta) = \frac{1}{2ik}\sum_{l=0}^{\infty}(2l+1)\left(e^{2i\delta_l}-1\right)P_l(\cos\theta) \tag{12}$$

This is the famous Faxen-Holtzmark formula. The values of the partial amplitudes following from it are:

$$f_l = \frac{1}{2ik}\left(e^{2i\delta_l}-1\right) \tag{13}$$

The partial cross-section (see eq. (3)) reads:

$$\sigma_l = \frac{4\pi}{k^2}(2l+1)\sin^2\delta_l \tag{14}$$

Appendix. The representation of the plane wave by partial spherical waves and its asymptotics.

Any cylindrically symmetric solution $\psi_0(r,\theta)$ of the stationary SE for free particle regular at origin can be represented as a series:

$$\psi_0(r,\theta) = \sum_{l=0}^{\infty} A_l j_l(kr) P_l(\cos\theta), \tag{15}$$

In our special case $\psi_0(r,\theta) = e^{ikr\cos\theta}$. On the other hand any function $g(\theta)$ on the interval $(0,\pi)$ can be represented by the Fourier-Legendre series:

$$g(\theta) = \frac{1}{2}\sum_{l=0}^{\infty}(2l+1) g_l P_l(\cos\theta), \tag{16}$$

where $g_l = \int_0^\pi g(\theta) P_l(\cos\theta)\sin\theta d\theta$. For the plane wave $g(\theta) = e^{ikr\cos\theta}$ equation (15) shows that $g_l = A_l j_l(kr)$. It means that the following integral representation is correct:

$$j_l(z) = \frac{2l+1}{2} A_l^{-1} \int_{-1}^{1} e^{izx} P_l(x)\, dx \tag{17}$$

To prove it and to find the coefficient A_l we use the Rodrigues formula:

$$P_l(x) = \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l \left(x^2-1\right)^l \tag{18}$$

and perform integration by part in the integral (17) l times. The result is:

$$j_l(z) = \frac{2l+1}{2}\frac{A_l^{-1}(iz)^l}{2^l l!}\int_{-1}^{1} e^{izx}\left(1-x^2\right)^l dx \tag{19}$$

In the course QM1 it was demonstrated that the function $\chi_l(z) = \frac{j_l(z)}{z^l}$ satisfies the following equation:

$$z\frac{d^2\chi_l}{dz^2} + \frac{d\chi_l}{dz} + z\chi_l = 0 \tag{20}$$

Please, check that the integral in the right-hand side of eq. (19) staisfies this equation. Thus, we have proved that the right-hand side of eq. (19) is proportional to $j_l(z)$. They will be equal if A_l is properly chosen. Compare the values of χ_l and the value $\left(2^l l!\right)^{-1} \int_{-1}^{1} \left(1-x^2\right)^l dx$ and $\chi_l\left(0\right)$. The integral can be easily calculated. with the xhange of variable $x^2 = t$. It is equal to the Euler beta function $B\left(l+1, \frac{1}{2}\right) = \frac{\Gamma(l+1)\Gamma\left(\left(\frac{1}{2}\right)\right)}{\Gamma\left(l+\frac{3}{2}\right)} = \frac{2^{l+1}l!}{(2l+1)!!} = \frac{2^{2l+1}(l!)^2}{(2l+1)!}$. Thus, $\left(2^l l!\right)^{-1} \int_{-1}^{1} \left(1-x^2\right)^l dx = \frac{2^{l+1}l!}{(2l+1)!}$. We have seen in the course QM 1 that $\chi_l\left(z\right) = \left(-\frac{1}{z}\frac{d}{dz}\right)^l \frac{\sin z}{z}$. The value of this function at $z = 0$ is equal to $(-1)^l 2^l \left(\frac{d}{dz^2}\right)^l \sum_{n=0}^{\infty} \left(-1\right)^n \frac{\left(z^2\right)^n}{(2n+1)!}|_{z=0} = \frac{2^l l!}{(2l+1)!}$. Thus, we find that $A_l = \left(2l+1\right) i^l$. Plugging these values in eq. (15) we get:

$$e^{ikr\cos\theta} = \sum_{l=0}^{\infty} \left(2l+1\right) i^l j_l\left(kr\right) P_l\left(\cos\theta\right) \tag{21}$$

Employing the asymptotic of the spherical Bessel function $j_l\left(kr\right) \cong \frac{\sin\left(kr-\frac{l\pi}{2}\right)}{kr}$, we arrive at eq. (8).