Course 624-08, Quantum Mechanics 2

Mathematical Appendix 2

Legendre Polynomials

a) *Legendre polynomials. Definition and generating function.*

The Legendre polynomials $P_l(x)$ are solutions of the Legendre differential equation:

$$\frac{d}{dx}\left[\left(1-x^2\right)\frac{dP_l}{dx}\right]+l\,(l+1)\,P_l=0, \tag{1}$$

regular at $x=\pm 1$. The explicit form of such a solution is given by Rodriguez formula

$$P_l(x)=\frac{1}{2^n n!}\left(\frac{d}{dx}\right)^l\left(x^2-1\right)^l \tag{2}$$

From this equation it follows that $P_l(x)$ are polynomials which are either even or odd function of x depending on parity of l and that for any l these polynomials accept value 1 at $x=1$. The generating function for the Legendre polynomials is $f(x,s)=\frac{1}{\sqrt{1+s^2-2xs}}$. We will show that:

$$f(x,s)=\sum_{l=0}^{\infty}s^l P_l(x) \tag{3}$$

To prove this formula, we consider a well-known solution of the Laplace equation $\frac{1}{|\mathbf{r}-\mathbf{r}'|}=\frac{1}{\sqrt{r^2+r'^2-2rr'\cos\theta}}=\frac{1}{r'}f\left(\cos\theta,\frac{r}{r'}\right)$, where θ is the angle between the vectors $\mathbf{r}$ and $\mathbf{r}'$. This the electric potential created in the point $\mathbf{r}$ by an unit charge place at the point $\mathbf{r}'$. As any solution of the Laplace equation it can be expanded in a superposition of a complete of the Laplace equation solutions:

$$\frac{1}{\sqrt{r^2+r'^2-2rr'\cos\theta}}=\sum_{l=0}^{\infty}a_l r^l P_l(\cos\theta) \tag{4}$$

To find coefficients a_l, let us consider equation (4) at $\theta=0$ ($\cos\theta=1$) and $r'>r$. Then the left-hand side of equation turns into $\frac{1}{r'-r}=\sum_{l=0}^{\infty}\frac{r^l}{r'^{l+1}}$. Thus, we find $a_l=\frac{1}{r'^{l+1}}$ at $r'>r$. From this result equation (3) directly follows.

b) *Recurrent relations*

One of consequences of equation (3) is the recurrence relation between Legendre polynomials. Differentiating both sides of equation (3) by s, we arrive at a following result:

$$-\frac{s-x}{\left(1+s^2-2sx\right)^{3/2}}=\sum_{l=0}^{\infty}l s^{l-1}P_l(x) \tag{5}$$

Multiplying both sides of this equation by $1+s^2-2sx$ and applying again the equation (3), one finds equating from both sides the coefficients at s^l:

$$(2l+1)\,xP_l(x)=(l+1)\,P_{l+1}(x)+lP_{l-1}(x) \tag{6}$$

c) *Asymptotics*

From equation (3) using the Cauchy theorem, we find integral representation for Legendre polynomials:

$$P_l(x) = \frac{1}{2\pi i} \oint \frac{ds}{s^{l+1}\sqrt{1+s^2-2sx}} \tag{7}$$

in which the contour must surround the point $s=0$. Let consider large values of l and $x=\cos\theta$ not too close to 1, so that $l\theta \gg 1$. Then the saddle point method can be applied to the integral (7). The integrand can be represented as $\exp(-\phi)$, where

$$\phi = l\ln s + \frac{1}{2}\ln\left(1+s^2-2sx\right) = l\ln s + \frac{1}{2}\ln\left(s-e^{i\theta}\right) + \frac{1}{2}\ln\left(s-e^{-i\theta}\right) \tag{8}$$

Zeros of the derivative $\frac{\partial\phi}{\partial s}$ should be found from equation:

$$\frac{l}{s} + \frac{1}{2\left(s-e^{i\theta}\right)} + \frac{1}{2\left(s-e^{-i\theta}\right)} = 0 \tag{9}$$

Since l is large, they should be close to the zeros of the denominator $s = x \pm \sqrt{x^2-1} = \exp(\pm i\theta)$. With precision $1/l$ we find the saddle points:

$$s_\pm = e^{\pm i\theta}\left(1-\frac{1}{2l}\right) \tag{10}$$

In these points the second derivatives occurs to be $\phi''_\pm \approx -2l^2 e^{\mp 2i\theta}$. According to general steepest descent (saddle point) method the initial contour must be deformed in such way that it passes at the saddle points along the direction whose angle to the real axis is $\arg\phi''_\pm/2 = \pm\theta + \frac{\pi}{2}$ to the real axis. Thus, the contour must be directed perpendicular to the vectors $e^{\pm i\theta}$. Since the initial contour was a closed circle around the origin, the deformed contour will pass the two saddle points in opposite directions. Thus, the contribution of the point s_+ reads:

$$\sqrt{\frac{\pi}{\phi''_+}}\left.\frac{1}{s^{l+1}\sqrt{\left(s-e^{-i\theta}\right)\left(s-e^{i\theta}\right)}}\right|_{s=s_+} \approx -\sqrt{\frac{\pi}{2l\sin\theta}}e^{-i\left(l+\frac{1}{2}\right)\theta - i\frac{\pi}{4}} \tag{11}$$

Together with the complex conjugated contribution from the point s_- it gives the asymptotic for Legendre polynomial at $l \gg 1$ and $l\theta \gg 1$:

$$P_l(\cos\theta) \simeq \sqrt{\frac{2}{\pi l\sin\theta}}\sin\left[\left(l+\frac{1}{2}\right)\theta + \frac{\pi}{4}\right] \tag{12}$$

At small angles $\theta \ll 1$ the asymptotic can be obtained directly from equation (1) rewritten in the variable θ instead of $x=\cos\theta$ if everywhere replace $\sin\theta$ by θ and replace $l(l+1)$ by l^2:

$$\frac{1}{\theta}\frac{d}{d\theta}\left(\theta\frac{dP_l}{d\theta}\right) + l^2 P_l = 0 \tag{13}$$

This is equation for the Bessel function of zero order. Thus, at $l \gg 1$, $\theta \ll 1$ and arbitrary $l\theta$ the asymptotic of the Legendre polynomials reads:

$$P_l\left(\cos\theta\right) \simeq J_0\left(l\theta\right) \tag{14}$$

The ranges of validity of the two asymptotics overlap at $\theta \ll 1$, $l\theta \gg 1$. At this conditions the two asymptotics give the same answer with precision of relatively small values.