Part I
Ising model

1 Definition and the basic properties

The Ising model (IM) was invented by Lenz. His student Ising has found the partition function of the 1d model and tried to solve 2d model but failed. The exact solution of 2d model was found by Onsager in 1942-1944. It is one of the fundamental results in statistical physics. The IM consist of spins $\sigma_x = \pm 1$ ascribed to sites of a lattice $x$. The energy of the spin configuration is determined by the sum:

$$H = -J \sum_{nn} \sigma_x \sigma_x'$$

(1)

where the summation is extended to all nearest neighbor pairs (bonds) in the lattice. This is the isotropic homogeneous IM. A simplest generalization is the homogeneous, but anisotropic IM. On a square lattice it is defined as follows:

$$H = -\sum_{i,j} \left[ J_h \sigma_{i,j} \sigma_{i,j+1} + J_v \sigma_{i,j} \sigma_{i+1,j} \right]$$

(2) Here $(i; j)$ is the vector number of a site ($i$ denotes the number of line, $j$ denotes the number of column). The IM can be defined on different lattices: square, hexagonal, regular triangular, Kagome etc. (see Fig.1) and also on multidimensional lattices. It also can be inhomogeneous, i.e. the coupling constants $J$ may be different for different bonds.

Fig. 1a. Square lattice
4 nearest neighbors

Fig. 1b. Hexagonal lattice.
3 nearest neighbors

Fig. 1c. Triangular lattice. 6 nn.

Fig. 1d. Kagome Lattice
4 nn.
In a simplest case of the isotropic homogeneous IM (Eq. 1) the ground state depends on
the sign of $J$. If it is positive, all the spins in the ground state have the same value, either
all $+1$ or all $-1$ (up or down). In both cases the energy of the ground state is equal to
$E_g = -N_s J$, where $N_s$ is the total number of bonds. This is ferromagnetic IM (FIM). If $J$
is negative, generally the situation is more complicated. However it is still simple for
lattices that can be divided into two sublattices $A$ and $B$ in such a way that each site of
the sublattice $A$ has nearest neighbors belonging to $B$ and vice versa and one sublattice
turns into another at a symmetry operation of the lattice (divisible lattices). The square
and hexagonal lattices satisfy these conditions, whereas the triangular and Kagome
lattices do not. The energy of any configuration of Ising spins on divisible lattices is
invariant with respect to changing sign of all coupling constants (even in the
inhomogenous case) and simultaneous change of sign of all spins in one of the two
sublattices. Thus, the spectrum of AFIM coincides with the spectrum of the FIM, but
corresponding states differ by the sign of spins of one of sublattices. Non-divisible
lattices with negative coupling constant $J$ have very degenerate ground state with finite
entropy. Indeed, considering an elementary triangle in regular triangular lattice, we see
that the minimal energy corresponds to two spins parallel each other and one antiparallel
to them, i.e. $E = -N_s |J|/3$. If two spins are already chosen to be antiparallel, the third
remains free. This freedom gives the entropy $S = \frac{1}{3} \ln 2$ per site (prove this) contradicting
to the Nernst theorem. Such a situation is called frustration.

The FIM has the long-range order (LRO) at zero temperature. It means that the average
value of the spin on a site is not zero. At very high temperature it is disordered. Indeed,
consider the correlation function of two spins in one line $G(l) = \langle \sigma_{i,j} \sigma_{i,j+l} \rangle$. It is
monotonously decreasing function of $l$. At $l = 1$, it is equal to the average spin on a bond.
The Boltzmann distribution gives for the average spin on a bond the value
$\langle s_i \rangle = \tanh \frac{J}{T} = \frac{1}{2}$. Each next step on the line changes the correlation function by the same
factor since the dominant correlation is to the nearest neighbor and even it is very small.
Therefore $G(l) \approx \left(\frac{1}{2}\right)^l$. It tends to zero when $l \to \infty$. Thus, there is the long-range order
at zero temperature that vanishes at large temperature. There are two opportunities: the
LRO may vanish at any arbitrarily small temperature, though on very large distances, or
it can vanish at a finite temperature $T_c$. To establish what happens let us consider thermal
excitations of the ground state. A simplest excitation is the flip of one spin. Then 4 bond
energies change sign for the square lattice (Fig. 2a). The energy of such an excitation is
$8J$.

A more general excitation is the spin reversal inside some area $A$ restricted by a
contour $C$ of the length $L$ (Fig. 2b). The energy of such excitation is $E(L) = 2LJ$. The
free energy of such a contour is $F(L) = E - TS(L)$. The entropy of the contour of the
length $L$ can be roughly estimated as $S(L) = L \ln 3$ assuming that after arriving at some
point the next unit peace of contour can occupy with the equal probability one of 3
positions (it can not go back). This is obviously overestimated value of entropy since the
contour cannot occupy any bond twice or more and the requirement of being closed also
in not satisfied. However, the real entropy per cite should not differ too much. Accepting
this estimate as a rough approximation, we see that the free energy turns into zero at a
temperature \( T_c = \frac{2J}{\ln 3} \) or equivalently \( 2K_c = \ln 3 \), where \( K = J / T \). Exact answer is \( K_c = \ln \left( 1 + \sqrt{2} \right) = \ln 2.41 \). It was obtained by Kramers and Wannier in 1940 (see the next section).

\[ Fig. 2a. \, \text{One spin-flip.} \quad \text{Fig. 2b. Domain and domain wall.} \]

2. **High Temperature and Low Temperature expansion. Dual transformation.**

High temperature expansion was proposed by a Dutch mathematician Van der Warden. The partition function for the Ising model reads:

\[
Z = \sum_{\{\sigma_x\}} \exp \left[ - \frac{H \{\sigma_x\}}{T} \right]
\]

where the summation proceeds over all possible configurations \( \{\sigma_x\} \) of Ising spins on sites and \( H \{\sigma_x\} \) is the value of energy (1) or (2) for the configuration \( \{\sigma_x\} \). For simplicity we consider the isotropic case and square lattice. Then the eq. (3) can be represented as follow:

\[
Z = \sum_{\{\sigma_x\}} \prod_b \exp (K\sigma_b) \quad (4)
\]

where \( K = J / T \), the product runs over all bonds and the spin on bond \( \sigma_b = \sigma_x \sigma_x' \) is the product of spins on the sites belonging to the bond. Let represent the exponent in the latter equation as follows:

\[
\exp (K\sigma_b) = \cosh K + \sigma_x \sinh K = \cosh \left( 1 + \sigma_b \tanh K \right) \quad (5)
\]

Plugging this expression in eq. (4), one arrives at the expansion in powers of \( z = \tanh K \):

\[
Z = 2^{N_x} \left( \cosh K \right)^{N_b} \sum_{l=0}^{N_b} q_l z^l \quad (6)
\]

where \( N_b \) is the number of bonds, \( N_x = N_b / 2 \) is the number of sites (for square lattice); the coefficients \( q_l \) are average values of corresponding clusters of \( l \) bonds:
\[ q_l = 2^{-N_b} \sum_{\{\sigma_i\}} \sum_{(b_1 b_2 \ldots b_l)} \left\langle \sigma_{b_1} \sigma_{b_2} \ldots \sigma_{b_l} \right\rangle \]  \tag{7}

The second summation in equation (7) runs over all possible sets of non-coinciding bonds, but after summation over all possible configurations of spins non-zero results give only sets in which bonds form closed contours. The reason is that each spin on site must enter zero, two or four times, otherwise summation over opposite projection of this spin gives zero. Thus, the coefficient \( q_l \) is equal to the number \( N_{c_l} \) of different closed contours on the square lattice (including disconnected ones) of the length \( l \). Now Eq. (6) can be rewritten in the following form:

\[ Z(K) = (\sqrt{2} \cosh K)^{N_b} \sum_{l=0}^{N_{c_l}(l)} N_{c_l}(l) \zeta^l, \]  \tag{8}

An important conclusion follows from Eqs. (6,8): there is no phase transition in any finite system. Indeed, the factor \( (\sqrt{2} \cosh K)^{N_b} \) is a regular function of \( K \) at any real \( K \). The remaining sum is a polynomial of a finite power \( N_b \) and therefore also is a regular function of \( K \). A singularity can appear only in the so-called thermodynamic limit \( N_b \to \infty \).

Now the construction of low-temperature expansion is in order. As we established earlier, the excitations are spin flips inside some finite area restricted by a closed contour of the length \( l \). The energy of such excitation is \( E(l) = 2Jl \). Thus, the partition function can be written as follows:

\[ Z(K) = e^{N_b K} \sum_{l=0}^{N_{c_l}(l)} N_{c_l}(l) \zeta^l, \]  \tag{9}

where \( \zeta = \exp(-2K) \). Note that the domain walls are contours made from the bonds of the lattice dual to the initial one, whose sites are centers of elementary cells of the initial lattice (in 2 dimensions). In the considered case of a simple square lattice the dual lattice is again a simple square lattice. Therefore, the numbers \( N_{c_l}(l) \) in equations (8) and (9) are identical. But it is not the case for the triangular lattice, whose dual lattice is hexagonal (honeycombs). Let introduce a dual variable \( \tilde{K} \) by the relationship:

\[ \exp(-2\tilde{K}) = \tanh \tilde{K} \]  \tag{10}

Comparing equations (8) and (9), one finds:

\[ Z(K) = Z(\tilde{K}) (\sqrt{2} \cosh \tilde{K})^{-N_b} e^{N_b K} \]  \tag{11}

The relation (10) transforms low-temperature (large \( K \)) into a high temperature (small \( \tilde{K} \)) and vice versa. It is easy to check that the relation (10) is mutual: \( \exp(-2\tilde{K}) = \tanh \tilde{K} \). Its explicitly symmetric form reads (derive all that):

\[ \sinh 2K \sinh 2\tilde{K} = 1 \]  \tag{12}

Kramers and Wannier assumed that there exists only one phase transition. Let the partition function \( Z(K) \) in the thermodynamic limit has only one singularity at \( K=K_c \). The equation (11) implies \( K_c = \tilde{K}_c \) and from eq. (12) one finds \( \sinh 2K_c = 1 \) or equivalently
Modification for the anisotropic IM is straightforward: the closed contours must be characterized by their horizontal and vertical lengths \( l_h \) and \( l_v \) and horizontal and vertical high-temperature variables \( z_{h,v} = \tanh K_{h,v} \) and low-temperature variables \( \zeta_h = \exp(-2K_v) \), \( \zeta_v = \exp(-2K_h) \) (note that the vertical bond of a domain wall separates the spins on the horizontal bond of the initial lattice and vice versa). Thus, the duality relation that replaces (10) on anisotropic lattice reads:

\[
\exp(-2K_h) = \tanh K_v \tag{13}
\]

The second relation \( \exp(-2K_v) = \tanh K_h \) follows from the first one eq. (13) and mutuality of the dual equation (10). Again, the assumption that there exists only one phase transition implies \( K_{h,v} = \bar{K}_{h,v} \). The phase transitions proceed at the line in the plane \((K_h, K_v)\) determined by equation:

\[
\sinh 2K_h \sinh 2K_v = 1 \tag{14}
\]

At fixed values of \( J_h, J_v \) the transition takes place at the crossing point of the curve (14) and the straight line \( K_h J_v - K_v J_h = 0 \) (Fig. 3). The dual transformation as in the isotropic case connects the values of partition function at low and high temperatures:

\[
Z(K_h, K_v) = \frac{e^{N_h K_h + N_v K_v}}{\left(\sqrt{2} \cosh K_h\right)^{N_h} \left(\sqrt{2} \cosh K_v\right)^{N_v}} Z(K_h, K_v) \tag{15}
\]

Employing the duality relations (13) and an analogous one with \( h \) and \( v \) permuted it is possible to rewrite eq. (15 in symmetric form:

\[
\left(\sinh 2K_h\right)^{N_v/4} \left(\sinh 2K_v\right)^{N_h/4} Z(K_h, K_v) = \left(\sinh 2K_h\right)^{N_h/4} \left(\sinh 2K_v\right)^{N_v/4} Z(K_h, K_v) \tag{16}
\]

In the next section the dual transformation will be derived in a more general context.

\[\text{Fig. 3. Phase transition line in the plane } K_h-K_v \text{ according to duality relation.}\]

3 Dual transformation and self-dual models

Here we consider the dual transformation in a class of lattice statistical models with the symmetry \( Z_N \) or \( C_N \) of integers \( n \) with respect to addition in \( \text{mod} \ N \) or the root \( \omega_n = \exp(i \frac{2\pi n}{N}) \) of the \( N \)-th power from 1 with respect to multiplication. They are called clock models. The statistical weight \( W_b \) of a lattice bond \( b \) depends on the variable \( \omega_b \)
that is equal to the product of $\omega_x \omega_x'$ on the sites belonging to the bond. Equivalently, $W_b$ depends on the variable $n_b = n_x - n_x'$ that is a discrete gradient field. Note that $W_b(n_b)$ may be different functions at different bonds. The IM is a special case of the clock model with $N = 2$. The partition function of the clock model can be represented as follows:

$$Z = \sum_{\{n_b\}} \prod_b W_b(n_b) \prod_p \delta_N \left( \sum_{b' \in p} n_{b'} \right)$$

(17)

The summation in Eq. (16) runs over all possible values of the variables $n_b$ independently for each bond $b$ with the statistical weight $W_b(n_b)$. The index $b$ at the statistical weight indicates that it may be different for different bonds. The $\delta$-functions enforce the circulation of the discrete vector field $n_b$ to be zero on each elementary plaquette $p$ and enable independent summation over all $n_b$. The centers of plaquettes form a lattice dual to the initial one (Fig. 4). For a square lattice the dual lattice is also a square lattice. But the lattice dual to a hexagonal one is regular triangular. It is obvious that duality of lattices is a mutual property. Each bond $b$ of the initial lattice crosses only one bond $\bar{b}$ of the dual lattice (see Fig. 4). These bonds can be considered as dual each other.

The purpose of the following calculation is to show that the partition function (16) can be represented as analogous partition function over the dual lattice. For this purpose it is necessary to use the explicit representation of delta-function over the modulus $N$:

$$\delta_N(k) = \frac{1}{N} \sum_{m=0}^{N-1} \exp \left( i \frac{2\pi km}{N} \right)$$

(18)

Let us substitute Eq. (17) into the expression for $Z (16)$ with $k = \sum_{b' \in p} n_{b'}$ for a plaquette $p$ and $m = m_p$ different and independent for each plaquette. First we perform summation over $n_b$ corresponding to a fixed bond $b$. The sum reads:

$$\frac{1}{N} \sum_{m=0}^{N-1} W_b(n_b) \exp \left( i \frac{2\pi n_b m_b}{N} \right) = \tilde{W}_b(m_b),$$
where \( m_b \) is a dual to \( n_b \) variable on a dual bond \( \tilde{b} \) determined as \( m_b = m_{b'} - m_p \) if the dual bond goes from the center of plaquette \( p \) to the center \( p' \). We see that the dually transformed statistical weight \( W_b(m_b) \) is the Fourier transform of the initial statistical weight \( W_b(n_b) \) and that the dual bond vector field \( m_b \) is again a discrete gradient. Thus, after the dual transformation we find:

\[
Z = \sum_{\{m_b\}} \tilde{W}(m_b) \prod_p \delta_N \left( \sum_{b \in \partial p} m_b \right) \tag{20}
\]

This transformation establishes statistical equivalence of two generally different models on different lattices. However, in some cases the dual statistical weights are the same function of the bond spin and differ only by parameters. If simultaneously the dual lattices are of the same type, the models are called self-dual. A simplest example of the self-dual model is the so-called \( N \)-state Potts model on a square lattice. The Potts model is defined as a clock model with the statistical weight on the bond \( W(n) = A \delta_{n,b} + B \)

where \( A \) and \( B \) are some constants. Its Fourier-transform is \( \tilde{W}(m) = \frac{A}{N} + B \delta_{m,0} \).

Introducing a ratio \( R = A/B \) as a characteristic of the statistical weight, we find \( R \tilde{R} = N \).

The phase transition proceeds at \( R = \tilde{R} = \sqrt{N} \). In the case of the isotropic IM (\( N = 2 \)), \( n = 0 \) corresponds to parallel spins and \( n = 1 \) to antiparallel. Therefore, \( A + B = e^K ; B = e^{-K} ; R = A/B = e^{2K} - 1 \). The duality relation reads \( (e^{2K} - 1)(e^{2\tilde{K}} - 1) = 2 \)

that is equivalent to \( e^{-2K} = \tanh \tilde{K} \). Please, analyze the duality relation for \( N = 3 \) positional clock model. Note that in 2 dimensions the model dual to a model of vector field on bonds belongs to the same class. In 3 dimensions such a dual model has variables from the group \( Z_N \) ascribed to a plaquette perpendicular to the initial bond, i.e. quite different topology.

### 4 Ising model with magnetic field. Lattice gas.

The magnetic field \( h \) can be introduced into the Ising model so that the total Hamiltonian becomes:

\[
H = -J \sum_b \sigma_b \sigma_{b'} - h \sum_x \sigma_x \tag{21}
\]

At non-zero magnetic field \( h \) the phase transition disappears even in the thermodynamic limit. The reason is that the magnetization appears at any temperature as a result of violation of the \( Z_2 \) symmetry by magnetic field. At zero magnetic field and \( T < T_c \) the magnetization takes one of two equilibrium values \( \pm m(T) \) that runs from 0 at \( T = T_c \) to \( \pm 1 \) at \( T = 0 \). The phase transition proceeds only at \( h = 0 \).

The variable \( \sigma_x = \pm 1 \) can be expressed in terms of \( Z_2 \) variable \( n_x = 0 ; 1 \) as \( \sigma_x = 1 - 2n_x \).

Then the Hamiltonian (20) turns into the Hamiltonian for the lattice gas:
where $U = 4J$ and $\mu = 4J + h$. The model can be interpreted as a model of particles with a short-range interaction (they interact only on nearest neighbor sites). Each site can be filled with a particle ($n_x = 1$) or empty ($n_x = 0$). The number of particles is conserved. The value $\mu$ is the chemical potential of the gas. The phase transition proceeds at $T = T_c$ and $\langle n \rangle = n_c = 1/2$ that corresponds to zero magnetization of the IM. At $T < T_c$ the average density takes one of two equilibrium values $n(T) = n_c \pm m(T)/2$. The smaller of them can be interpreted as the density of a gas, whereas the larger should be the density of a liquid in equilibrium with the gas. Thus, the line $n = 1/2$, $T < T_c$ is the line of the first order phase transitions or liquid-gas equilibrium. Its upper point $T = T_c$, $n = n_c = 1/2$ is the critical point liquid-gas. The model of Lattice gas possesses an additional symmetry of reflection with respect to $n = 1/2$ that the real gas-liquid system does not have. Nevertheless, it reproduces correctly the critical behavior near the critical point due to universality. It means that real critical point has the asymptotic symmetry of the IM.

5 Transfer-matrix.

Kramers and Wannier in the same famous work of 1940 proposed a new approach to the Ising problem: the transfer-matrix method. The idea is to consider one of directions in the lattice as discrete time. More accurately, let us consider the square IM with $M$ lines and $N$ rows and introduce the partition function $Z_M(\Theta)$ as function of a spin configuration $\Theta = (\sigma_1, \sigma_2, \ldots, \sigma_N)$ in the last line. There are $2^N$ independent spin configurations $\Theta$. The partition function with number of lines larger by one $Z_{M+1}(\Theta)$ is related to $Z_M(\Theta)$ by a linear equation:

$$Z_{M+1}(\Theta) = \sum_{\Theta'} T_{\Theta\Theta'} Z_M(\Theta')$$  \hspace{1cm} (23)

The matrix $T_{\Theta\Theta'}$ is called transfer-matrix. For the anisotropic 2d IM its matrix element reads:

$$T_{\Theta\Theta'} = \exp \left[ \sum_{j=1}^{N} \left( K_h \sigma_j \sigma_{j+1} + K_v \sigma_j \sigma_{j-1} \right) \right]$$  \hspace{1cm} (24)

The periodic boundary condition $\sigma_{j+N} = \sigma_j$ is used. Repeating this operation, one finds:

$$Z_M = T^M Z_0$$  \hspace{1cm} (25)

Identifying configurations $\Theta_0 = \Theta_M$ and performing summation over them, one arrives at the result:

$$Z = \text{Tr} T^M$$  \hspace{1cm} (26)

All matrix elements (23) are positive. They are not symmetric, but can be easily symmetrized by a minor modification of Eq. (23):

$$T_{\Theta\Theta'} = \exp \left[ \sum_{j=1}^{N} \left( \frac{1}{2} K_h \sigma_j \sigma_{j+1} + \frac{1}{2} K_v \sigma_j \sigma_{j-1} + K_h \sigma_j \sigma_{j+1} + K_v \sigma_j \sigma_{j-1} \right) \right]$$  \hspace{1cm} (27)

A real symmetric matrix can be diagonalized by an orthogonal transformation. A theorem
of algebra states that such a matrix with all positive matrix elements has only positive eigenvalues. Let the transfer-matrix eigenvalues are equal to \( t_1, t_2, \ldots, t_N \). Eq. (25) can be rewritten as follows:

\[
Z = \sum_{j=1}^{2^N} (t_j)^M \tag{28}
\]

In the thermodynamic limit \( M \to \infty \), the contribution of the maximal eigenvalue in this sum is dominant, others can be neglected. Thus, in the thermodynamic limit

\[
Z = (t_{\max})^M \tag{29}
\]

and the Helmholtz free energy reads:

\[
F = -T \ln Z = -MT \ln t_{\max} \tag{30}
\]

The problem is reduced to calculation of the maximal eigenvalue of the transfer-matrix.

**Problem** Find the free energy and magnetization of one-dimensional IM in external magnetic field.

**6 Onsager solution.**

The transfer-matrix (24) can be represented as product of two matrices \( T = T_2 T_1 \), where \( T_1 \) is diagonal matrix with matrix elements:

\[
(T_1)_{\theta \theta'} = \delta_{\theta \theta'} \exp \left( \sum_{j=1}^{N} \sigma_j \sigma_{j+1} \right) \tag{31}
\]

and non-diagonal matrix \( T_2 \) has matrix elements:

\[
(T_2)_{\theta \theta'} = \exp \left( \sum_{j=1}^{N} \sigma_j \sigma_{j}' \right) \tag{32}
\]

The matrix \( T_2 \) can be represented as direct product \( T_2 = \otimes_{j=1}^N \gamma_j \) of \( N \) matrices of the same form acting in 2-dimensional subspaces \( R_j \):

\[
\gamma_j = \begin{pmatrix} e^{K_i} & e^{-K_i} \\ e^{-K_i} & e^{K_i} \end{pmatrix} = e^{K_i} \left( I + e^{-2K} \sigma_j \right), \tag{33}
\]

where \( \sigma_j \) are Pauli matrices acting in the space \( R_j \). Employing the dual transformation, the same matrix can be represented as \( \gamma_j = \sqrt{2 / \sinh 2K h} e^{K_i} \sigma_j \). Thus, the matrix \( T_2 \) can be rewritten as follows:

\[
T_2 = \left( \frac{2}{\sinh 2K h} \right)^{N/2} \exp \left( \sum_{j=1}^{N} K \sigma_j \right) \tag{34}
\]

The matrix \( T_1 \) acting in the same \( 2N \) dimensional space \( R_1 \otimes R_2 \otimes \ldots \otimes R_N \) can be represented also in terms of the Pauli matrices:
The dual transformation to a new set of Pauli matrices is defined on half-integer (dual) sites as follows:

\[
\tau^x_{j+1/2} = \sigma^x_j \sigma^{z+1}_j; \quad \tau^z_{j+1/2} = \prod_{k=1}^{j} \sigma^z_k
\]

(36)

at a constraint \(\tau^z_{1/2} = 1\), interchange \(T_1\) and \(T_2\). Though they do not commute, the partition function (26) being a trace does not change except of the preexponential factor in \(T_2\) (34). Thus we find that the dual transformation does not change the partition function with precision of a regular function:

\[
Z(K_h, K_v) = (\sinh 2K_h \sinh 2K_v)^{N/2} Z(\tilde{K}_h, \tilde{K}_v)
\]

(37)

For further calculation it is convenient to interchange \(z\) and \(x\) in the transfer-matrix so that

\[
T_1 = \exp \left( K_h \sum_{j=1}^{N} \sigma^x_j \sigma^{z+1}_j \right); \quad T_2 = (2 \sinh 2K_v)^{N/2} \exp \left( \tilde{K}_h \sum_{j=1}^{N} \sigma^z_j \right)
\]

(38)

Next we apply Wigner-Jordan transformation from spin to Fermi operators:

\[
a_j = \frac{1}{2} \prod_{k=1}^{j-1} \sigma^z_k \sigma^+_j; \quad a^+_j = \frac{1}{2} \prod_{k=1}^{j-1} \sigma^z_k \sigma^-_j,
\]

(39)

where \(\sigma^\pm_j = \sigma^x_i \pm i \sigma^y_j\). It is easy to check that operators \(a_j\) and \(a^+_j\) anticommute at different sites and obey a system of anticommuting relations:

\[
\{a_j, a_j\} = 0; \quad \{a^+_j, a^+_j\} = 0; \quad \{a_j, a^+_j\} = \delta_{jj'}
\]

(40)

From Eqs. (39) and (40) it follows:

\[
\sigma^z_j = 1 - 2a^+_j a_j; \quad \sigma^+_j \sigma^-_{j+1} = (a_j + a^+_j) \sigma^z_j (a_{j+1} + a^+_j) = (a^+_j - a_j) (a_{j+1} + a^+_j)
\]

(41)

Plugging these equations into (38), we find the fermion representation of the operators \(T_1, T_2\):

\[
T_1 = \exp \left[ K_h \sum_{j=1}^{N} (a^+_j - a_j) (a_{j+1} + a^+_j) \right]
\]

(42)

\[
T_2 = (2 \cosh K_v)^{N} \exp \left( -2 \tilde{K}_h \sum_{j=1}^{N} a^+_j a_j \right)
\]

(43)

It is natural to use the Fourier-transformed operators
The eigenvalues of the product of these two matrices are: 1. Therefore, numbers. For each operator and its corresponding operator, the fermion occupation numbers in the states spanned onto states are eigenstates with the operator \( \Gamma = \sigma_{j}^{z} \sigma_{j}^{z} \sigma_{N}^{z} \). This operator commutes with the transfer-matrix since it anticommutes with any \( a_{j} \) and \( a_{j}^{\dagger} \). Therefore, all eigenstates of the transfer-matrix are characterized either by \( \Gamma = 1 \) or \( \Gamma = -1 \). On the other hand, \( a_{j+N} = \sigma_{j}^{z} \sigma_{j}^{z} \sigma_{N}^{z} \sigma_{j}^{z} \sigma_{j}^{z} \sigma_{j} \). Here we have used the periodic boundary conditions for \( \sigma_{j}^{z} \) and \( \sigma_{j}^{\pm} \) and anticommutation of any \( a_{j} \) and \( \Gamma \). Thus, for even states \( a_{j+N} = a_{j} \) and for odd states \( a_{j+N} = -a_{j} \). The discrete values of momentum are \( p_{k} = \frac{\pi (2k - 1)}{N} \) and \( p_{k} = \frac{2 \pi k}{N} \) for even and odd states, respectively. In both cases the matrices \( T_{1} \) and \( T_{2} \) in Fourier-representation are factorized to the product of matrices \( T_{1p} \) and \( T_{2p} \) \( (p > 0) \), where with precision of regular factors:

\[
T_{1p} = \exp \left[ 2K_{h} \left( a_{p}^{\dagger} a_{p} + a_{-p}^{\dagger} a_{-p} - 1 \right) \cos p + i \left( a_{p}^{\dagger} a_{p}^{\dagger} - a_{-p} a_{-p} \right) \sin p \right] \quad (44)
\]

\[
T_{2p} = \exp \left[ -2K_{h} \left( a_{p}^{\dagger} a_{p} + a_{-p}^{\dagger} a_{-p} \right) \right] \quad (45)
\]

Thus, the problem is reduced to the diagonalization of the matrices in a four-dimensional space spanned onto states \( |0,0\rangle, |1,0\rangle, |0,1\rangle \) and \( |1,1\rangle \), where the numbers 0 and 1 denote the fermion occupation numbers in the states \( p \) and \( -p \). This problem is further reduced to diagonalization in 2-dimensional subspaces spanned onto pairs of vectors \( |0,0\rangle, |1,1\rangle \) and \( |1,0\rangle, |0,1\rangle \) since all operators in \( T_{1p} \) and \( T_{2p} \) conserve the parity of occupation numbers. For each operator \( T_{1p} \) and \( T_{2p} \) the states \( |1,0\rangle, |0,1\rangle \) are eigenstates with the eigenvalue 1. Therefore, only the remaining 2-d subspace spanned on the states \( |0,0\rangle, |1,1\rangle \) should be considered. In this subspace the operator \( a_{p}^{\dagger} a_{p} + a_{-p}^{\dagger} a_{-p} - 1 \) acts as \( \sigma_{x} \) and the operator \( i \left( a_{p}^{\dagger} a_{p}^{\dagger} + a_{-p} a_{-p} \right) \) acts as \( -\sigma_{y} \). Multiplying one of this state by a phase factor \( i \), it is possible to replace \( -\sigma_{y} \) by \( \sigma_{x} \). Thus, the equivalent form of the matrices (43,44) reads:

\[
T_{1p} = \exp \left[ 2K_{h} \left( \sigma_{x} \cos p + \sigma_{z} \sin p \right) \right] = \cosh 2K_{h} + \left( \sigma_{x} \cos p + \sigma_{z} \sin p \right) \sinh 2K_{h} \quad (46)
\]

\[
T_{2p} = \exp \left( -2K_{h} \sigma_{z} \right) \quad (47)
\]

or in the matrix form:

\[
T_{1p} = \begin{pmatrix}
\cosh 2K_{h} + \sinh 2K_{h} \cos p & \sinh 2K_{h} \sin p \\
\sinh 2K_{h} \sin p & \cosh 2K_{h} - \sinh 2K_{h} \cos p
\end{pmatrix} \quad (48)
\]

\[
T_{2p} = \begin{pmatrix}
e^{-2K_{h}} & 0 \\
0 & e^{2K_{h}}
\end{pmatrix} \quad (49)
\]

The eigenvalues of the product of these two matrices are:
\[ t_p^± = \cosh 2K_h \cosh 2\tilde{K}_h - \sinh 2K_h \sinh 2\tilde{K}_h \cos p \pm \varepsilon_p, \]  

(50)

where

\[ \varepsilon_p = \left[ \cosh 2K_h \cosh 2\tilde{K}_h - \sinh 2K_h \sinh 2\tilde{K}_h \cos p + \left( \sinh 2K_h \sin p \right)^2 \right]^{1/2} \]  

(51)

It is obvious that \( t_p^+ > 1 > t_p^- \). Therefore, the maximum eigenvalue of the total transfer-matrix is:

\[ t_{\text{max}} = \prod_p t_p^+ \]  

(52)

The Helmholtz free energy reads:

\[ F = -TM \ln t_{\text{max}} = -TM \sum_p \ln t_p^+ \]  

(53)

In the thermodynamic limit \( N \to \infty \) the sum in Eq. (51) turns into integral:

\[ F = -TM \int_{-\pi}^{\pi} \frac{dp}{2\pi} \ln t_p^+ \]  

(54)

The integral has singularity at \( K_h = \tilde{K}_h \). At this relation fulfilled and \( p = 0 \), the expression under square root turns into zero. To elucidate the character of singularity, let us estimate \( t_p^+ \) at \( |K_h - \tilde{K}_h| \ll K_h \) and \( p \ll 1 \):

\[ t_p^+ = 1 + \sqrt{4 \left( K_h - \tilde{K}_h \right)^2 + \left( \sinh 2K_h \right)^2 p^2} \]  

(55)

Thus, near the singularity the following approximation is valid:

\[ \ln t_p^+ = \sqrt{\tau^2 + \left( \sinh 2K_h \right)^2 p^2}, \]  

(56)

where the value \( \tau = 2 \left( K_h - \tilde{K}_h \right) \) can be accepted as a temperature deviation from the phase transition point. Then the specific heat near phase transition is:

\[ C = -\frac{MN}{\pi \sinh 2K_h} \ln \tau \]  

(57)

This is singular behavior, the famous Onsager’s result. Still the physics behind the singularity is not visible. Who are the fermions playing so important role in the problem also is not clear. The second critical exponent of the magnetization requires even more difficult calculations. Onsager first presented it at a Conference in 1948 without derivation, four years after his large article was published and 6 years after his first short communication in Phys. Rev. Letters. C.N. Yang published the derivation of this second Onsager great achievement in 1952 and it took about 20 pages in a Phys. Rev. article. It shows that a simpler approach with more visible physics is much appreciated. Such an approach will be presented in the next section.
7 Strongly anisotropic IM. Domain walls as fermions.

In this section we will show that the fermions can be visualized as domain walls. We will derive their Hamiltonian and obtain the correlation length and singularity of specific heat from it. We also will find the critical exponent \((1/8)\) for the magnetization.

In a strongly anisotropic IM \((J_h \ll J_v)\) the energy per unit length of the horizontal domain wall (DW) is much larger than that of the vertical one. Therefore, the DW at temperatures between \(J_h\) and \(J_v\) in average have long vertical pieces and short horizontal ones. Since the statistical weight of the unit interval is exponentially small \(\exp(-2K_v)\), the probability of longer horizontal intervals can be neglected. On the other hand, the contribution to the "action" (exponent in statistical weight) of a unit vertical interval in the same range of temperatures is \(2K_h \ll 1\). Therefore, the discretness of vertical coordinate ("time") can be neglected, whereas the discretness of the horizontal coordinate still remains substantial at any temperature except of a small vicinity of the critical temperature \(T_c\). We will show that it is located in the same range of temperature: \(J_h \ll T_c \ll J_v\). Everywhere in the chosen range of temperature the DW propagate dominantly in the vertical direction. The unit horizontal intervals appear on the average distance \(\exp(2K_v)\) directed with equal probability left or right.

There are no crossings of two DWs in one point. However, two domain walls that propagate vertically at distance 1 can annihilate and two domain walls can appear at some “moment of time” at the unit distance each to other (Fig. 5). Therefore it is possible to interpret the DWs as world lines of fermions whose number is not conserved. The Hamiltonian for these fermions is a generator of "time" evolution. Elementary processes included in the evolution on the next step are shown on Fig. 5. The statistical weight of a step without displacement of coordinate is \(2K_h\); for a step with horizontal displacement the statistical weight is \(-\exp(-2K_v) = -\tilde{K}_h\). The same statistical weight can be ascribed to the processes of creation and annihilation of a pair of domain wall lines closing the domain wall loops. Thus, the Hamiltonian of the DW as fermions reads:

\[
H = 2K_h \sum_{i=1}^{N_h} a_i^\dagger a_i - 2K_v \sum_{i=1}^{N_h} \left[ (a_{i+1}^\dagger + a_{i-1}^\dagger) a_i + a_{i+1} a_i^\dagger + a_i a_{i+1} \right] \tag{58}
\]

The Hamiltonian (55) can be reduced to a set of independent Hamiltonians \(H_p\) for pairs of opposite momenta by the Fourier transform:

\[
a_j = \frac{1}{\sqrt{N_h}} \sum_p a_p e^{ipj}; a_j^\dagger = \frac{1}{\sqrt{N_h}} \sum_p a_p^\dagger e^{-ipj} \tag{59}
\]
Fig. 5. Domain walls as fermions. The closed domain loop is treated as the world line of a fermion. Elementary steps on trajectory correspond to indicated parts of the Hamiltonian.

Then the Hamiltonian $H_p$ reads:

$$H_p = 2\left(K_h - \tilde{K}_h \cos p\right)(a_p^\dagger a_p + a_{-p}^\dagger a_{-p}) + 2i\tilde{K}_h \sin p\left(a_p^\dagger a_{-p} - a_{-p} a_p\right)$$  \hspace{1cm} (60)

By a simple phase transformation $\alpha_p \rightarrow e^{-i\pi/4} \alpha_p; \quad \alpha_p^\dagger \rightarrow e^{i\pi/4} \alpha_p^\dagger$, the same Hamiltonian $H_p$ takes the following form:

$$H_p = 2\left(K_h - \tilde{K}_h \cos p\right)(a_p^\dagger a_p + a_{-p}^\dagger a_{-p}) + 2i\tilde{K}_h \sin p\left(a_p^\dagger a_{-p} + a_{-p} a_p\right)$$  \hspace{1cm} (60')

In order to diagonalize this Hamiltonian we apply the Bogoliubov transformation:

$$a_p = u_p \alpha_p + v_p \alpha_{-p}^\dagger; \quad a_{-p}^\dagger = -v_p \alpha_p + u_p \alpha_{-p}^\dagger,$$  \hspace{1cm} (61)

with real coefficients $u_p, v_p$ satisfying the requirements of unitarity $u_p^2 + v_p^2 = 1$ and orthogonality $u_{-p} = u_p, \quad v_{-p} = -v_p$. Then $\alpha_p, \alpha_{-p}^\dagger$ satisfy canonical anticommutation relations: $\{\alpha_p, \alpha_{-p}'\} = \{\alpha_p^\dagger, \alpha_{-p}^\dagger\} = 0; \quad \{\alpha_p, \alpha_{-p}^\dagger\} = \delta_{pp'}$. Plugging the Bogoliubov transformation (61) into the Hamiltonian (60') and requiring the coefficient at $\alpha_p \alpha_{-p}$ to be zero, one obtains:

$$(K_h - \tilde{K}_h \cos p)2u_p v_p = \tilde{K}_h \sin p\left(u_p^2 - v_p^2\right)$$  \hspace{1cm} (62)

Together with the unitarity condition $u_p^2 + v_p^2 = 1$ it leads to the following result:

$$u_p^2 = \frac{1}{2} \left(1 + \frac{\xi_p}{\kappa_p}\right); \quad v_p^2 = \frac{1}{2} \left(1 - \frac{\xi_p}{\kappa_p}\right)$$  \hspace{1cm} (63)
where
\[ \xi_p = K_h - \tilde{K}_h \cos p; \quad \epsilon_p = \sqrt{\xi_p^2 + \tilde{K}_h^2 (\sin p)^2} \] (64)

The Hamiltonian expressed in terms of the transformed operators reads:
\[ H = E_G + \sum_p \epsilon_p \alpha_p^\dagger \alpha_p \] (65)

The ground state energy \( E_G \) is equal to
\[ E_G = -\sum_p (\epsilon_p - \xi_p) \] (66)

In the thermodynamic limit \( N, M \to \infty \) the ground state energy reads:
\[ E_G = -N \int_{-\pi}^{\pi} \frac{dp}{2\pi} (\epsilon_p - \xi_p) \] (67)

The phase transition proceeds at \( K_h = \tilde{K}_h \). In the phase transition point the fermion mass turns into zero as \( K_h - \tilde{K}_h \) and the correlation length turns to infinity as \( (K_h - \tilde{K}_h)^{-1} \). It shows that the critical exponent of the correlation length \( \nu = 1 \). The free energy can be calculated according to general equation (52) as:
\[ F = -T \alpha E_G = TMN \int_{-\pi}^{\pi} \frac{dp}{2\pi} (\epsilon_p - \xi_p) \] (68)

Near critical point its singular part behaves as \( -\tau^2 \ln|\tau| \), where \( \tau = K_h - \tilde{K}_h \). Thus, we reproduce the Onsager’s result on specific heat singularity.

It is straightforward to calculate the correlation function of fermions in critical point. First of all, note that exactly at critical curve \( K_h = \tilde{K}_h \) and at small \( p \ll 1 \) the Bogoliubov coefficients obey the relations \( u_p^2 = v_p^2 = \frac{1}{2} \). Therefore, the simultaneous correlation functions at large distances \(|x - x'| \gg 1\) are:
\[ \langle a_x^\dagger a_{x'} \rangle = \frac{1}{N} \sum_p e^{-ip(x-x')} v_p^2 = \frac{1}{2N} \sum_p e^{-ip(x-x')} = \frac{1}{2} \delta(x - x') \] (69)
\[ \langle a_x^\dagger a_{x'} \rangle = \langle a_x a_{x'} \rangle = \frac{i}{N} \sum_p e^{-ip(x-x')} u_p v_p = \frac{i}{2N} \sum_p e^{-ip(x-x')} \text{sign}p = \frac{1}{2\pi (x - x')} \] (70)

The last result was obtained in the thermodynamic limit that allows transition from summation to integration according to the rule \( \frac{1}{N} \sum_p e^{i\phi} \frac{dp}{2\pi} \). Since only small \( p \) matters the limits in the integral can be replaced by \( \pm \infty \) and \( u_p v_p \) by \( \frac{i}{2} \text{sign}(p) \).

For the calculation of the magnetization critical exponent the key observation is that the product of two Ising variables in one line \( \sigma_x \sigma_{x'} \) is equal to \( (-1)^{N_{dw}(x,x')} \), where \( N_{dw}(x,x') \) is the number of domain walls crossing the line (of constant "time") between \( x \) and \( x' \). According to our interpretation, it is the number of fermions \( N_f(x,x') \) simultaneously located in the interval of coordinate \((x,x')\):
\[ N_f(x,x') = \sum_{y=x}^{x'} a^+_y a_y \]  \hspace{1cm} (69)

The average of the product \( \sigma_x \sigma_{x'} \) can be written as:

\[
\langle \sigma_x \sigma_{x'} \rangle = \left\langle e^{i\pi N_{dw}(x,x')} \right\rangle = e^{i\pi N_{dw}(x,x')} \exp \left[ -\frac{\pi^2}{2} \left( \tilde{N}_{dw}(x,x') \right)^2 \right] \hspace{-0.2cm} \]  \hspace{-0.2cm} (70)

Here \( \tilde{N}_{dw}(x,x') = N_{dw}(x,x') - \langle N_{dw}(x,x') \rangle \) is the fluctuating part of domain wall number. In equation (70) we have used the law of large numbers assuming that \( \langle N_{dw}(x,x') \rangle \gg 1 \) and the fluctuation of the number of domain walls is Gaussian distributed. Thus the problem is reduced to the calculation of the average:

\[
\langle N_{dw}^2 \rangle = \sum_{x,y'\geq x} (a^+_y a_y a^+_y a_y) = \sum_{x,y'\geq x} \left[ \langle a^+_y a_y \rangle \langle a^+_y a_y \rangle - \langle a^+_y a_y \rangle \langle a^+_y a_y \rangle \right] \hspace{1cm} (71)
\]

We have used the Wick theorem according to which the average of product of Fermi-operators can be represented as sum of all possible their pairing taken with the sign of a permutation of these operators that was necessary to put the operators from their initial positions in the product to the positions they took at a pairing. Since we are interested in long distances, we replace the summation over \( y \) and \( y' \) by integration. For the averages we will use their critical point values (67, 68). The first term in the square brackets of eq. (71) is formally equal to

\[
\frac{1}{2} \int_x^{x'} dy \int_{x'}^y \left[ \delta(y-y') \right]^2 \hspace{1cm} (72)
\]

Square of the delta-function is not well-defined. In reality what we consider as \( \delta - \) function is a function that has a sharp peak with the width about the lattice constant where our approximation fails. Therefore, let us put one of \( \delta - \) functions equal to \( 1/a \), where \( a \) is the cut-off length (of the order of lattice constant), and integrate the second \( \delta - \) function. The result is \( (x' - x)/4a \): Employing Eq. (68), we find for the integral of the second term:

\[
-\frac{1}{4\pi^2} \int_x^{x'} dy \int_{x'}^y \frac{1}{(y-y')^2} = \frac{x - x'}{4a} - \frac{1}{4\pi^2} \int_x^{x'} \left( \frac{2}{b - y - x} - \frac{1}{y' - y} \right) \hspace{1cm} (73)
\]

The integral in Eq. (70) diverges at \( y = y' \). For regularization the integration over \( y' \) will be performed from \( x \) till \( y - b \) and from \( y + b \) till \( x' \). The result reads:

\[
\left\langle N_{dw}^2 \right\rangle = \frac{x - x'}{4a} - \frac{1}{4\pi^2} \int_x^{x'} \left( \frac{2}{b - y - x} - \frac{1}{x' - y} \right) \hspace{1cm} (74)
\]

The first linear in \( x - x' \) term in eq. (74) is compensated by the term independent on \( y \) under the integral at \( b = 2a/\pi^2 \). The linear term in the exponent would mean the finite correlation length, whereas we know that the correlation length at the phase transition point becomes infinite. However, as it is seen from our calculation, the adjustment of a constant at linear term in the exponent proceeds on short distances and therefore is beyond of precision of our long-scale approximation. In the exact Onsager’s solution this
adjustment proceeds automatically. The remaining terms after integration give logarithms diverging on the ends of the integration interval. This divergence is spurious since and is the result of the failure of continuous approximation on the scale of the lattice constant. It must be removed by the integration cut-off on the scale of lattice constant (unity). Thus, we find:

\[ \left\langle \left( \tilde{N}_{dw}(x,x') \right)^2 \right\rangle = \frac{1}{2\pi^2} \ln |x'| - x | \]  

(75)

Substituting it into the expression for the spin correlation function (70) we find:

\[ \langle \sigma_x \sigma_x' \rangle = e^{i \langle N_{dw}(x,x') \rangle} |x'| - x |^{-1/4} \]  

(76)

Thus, we have found a new critical exponent \(1/4\) for decay of the spin correlation in critical point. However, equation (76) contains also the oscillating exponential factor \(e^{i \langle N_{dw}(x,x') \rangle}\). Indeed, \(\langle N_{dw}(x,x') \rangle = \langle a_x^\dagger a_x \rangle (x' - x) = \langle n_{dw} \rangle \times (x' - x)\), where \(n_{dw}\) is the domain wall density whose average value remains finite (about 1/3) in the critical point. In reality these oscillations are spurious. They appear as a consequence of application of the long-wave approximation to the short scale where our approximation fails. Indeed the oscillations proceed on the scale \(\langle n_{dw} \rangle^{-1} \sim 3\). The value \(\langle n_{dw} \rangle\) with overwhelming probability is irrational. Therefore, the average phase from this “wave-vector” in the integer points of the lattice is zero by the modulus \(2\pi\). On the other hand, the power-like decay of the correlation amplitude is a consequence of the long-range fluctuations where the applied approximation is valid. The function \(1/4 \ln |x - x'|\) is slowly varying and is not sensitive to the discreteness of the lattice.

The physical phenomenon behind this phenomenon is as follows. Each spin creates near itself an effective field favoring the same polarization of neighbors. It therefore favors a longer domain of parallel spins near itself.

### 8 Scaling and other critical exponents

Since the correlation length turns into infinity in the critical point, there appears scale invariance or scaling of critical phenomena. We have seen already that the correlation length is inverse proportional to the deviation of temperature from critical \(R_c \propto \tau^{-1}\). We also have seen that the correlation of two spins is proportional to the length in power -1/4. It means that spin (or magnetization) varies with the length scale as \(L^{-1/8}\). Therefore, at temperature slightly below the phase transition, the spontaneous magnetization \(m(\tau)\) varies with temperature as \((-\tau)^{1/8}\). This result can be obtained by direct calculation from the domain-fermion representation. The spin correlation function can be obtained as we did before, but near the transition point the fermion correlation functions (67, 68) are no more power-like. They acquire an additional exponential factor \(e^{i \varphi - \frac{q R_c}{4}}\) as a consequence of finite “mass” of “relativistic” quasi-particles \(\varepsilon_p = \sqrt{m^2 + p^2}\) with \(m = (K_h - \tilde{K}_h) = |\tau|\). Therefore the logarithmic integration now in eq. (74) is effectively cut off not only at short, but also at long distances at the scale \(R_c = m^{-1}\).
Let us use scaling to calculate the critical behavior of the magnetic susceptibility $\chi$ on temperature near phase transition. The magnetic susceptibility is the derivative of magnetization by magnetic field:

$$\chi = \frac{dm}{dh}$$

(77)

From eq. (21) it follows that

$$m = \langle \sigma_x \rangle = T \frac{1}{NM} \frac{\partial (\ln Z)}{\partial h}$$

(78)

Differentiating this equation one more time, we find:

$$\chi = T \frac{\partial^2 \ln Z}{NM \partial h^2} = \frac{1}{NM} \sum_{x,x'} \langle \sigma_x \sigma_{x'} \rangle$$

(79)

In this sum $x$ and $x'$ denote points in 2d space rather on line. Exactly in the critical point this sum diverges since the correlation of two spins decays too slowly with distance between them. Beyond the critical point, but close to it the integration is effectively cut at the distance $|x' - x| \sim R_c \sim \tau^{-1}$. Thus

$$\chi \sim R_c^{7/4} |\tau|^{-7/4}$$

(80)

Simultaneously we have found that the magnetic field, as it follows from eq. (77) scales as $m / \chi \sim L^{−1/8−7/4} = L^{−15/4}$ or as $|\tau|^{15/4}$. The reader can try to apply these ideas to find how the magnetization behaves on magnetic field at $T = T_c$ or, equivalently, at $|\tau| \ll h$.

Another interesting possibility is to find how the finite size of system smears the phase transition. It happens when the correlation length becomes of the same order of magnitude as the size of system. For example, the specific heat will have a finite maximum instead of singularity. How does it depend on the size of system? At what deviation from critical temperature it stops to grow? If you have addressed this question, try to do the same for susceptibility.