Note on Born approximation: conditions of validity

The Born approximation for the scattering amplitude $f(\mathbf{n}, \mathbf{n}')$ is based on general equation:

$$f(\mathbf{n}, \mathbf{n}') = -\frac{m}{2\pi\hbar^2} \int e^{-ik\mathbf{n}'\mathbf{x}} V(\mathbf{x}) \psi^+(\mathbf{x}) d^3x \qquad ((1))$$

and substitution of the exact solution $\psi^+(\mathbf{x})$ by the incident wave $e^{i\mathbf{k}_i\mathbf{x}}$. Let us analyze when this approximation is valid. The exact wave function obeys an integral equation:

$$\psi^+(\mathbf{x}) = e^{i\mathbf{k}_i\mathbf{x}} - \frac{m}{2\pi\hbar^2} \int \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} V(\mathbf{x}') \psi^+(\mathbf{x}') d^3x' \qquad ((2))$$

The question is: at what condition the integral term is small in comparison to the incident wave? Let consider first the case when $ka \lesssim 1$, where a is a characteristic radius of the potential. The integration in (1) and (2) is limited by the potential in a volume with the linear size a, the exponent $e^{ik|\mathbf{x}-\mathbf{x}'|}$ in this region is of the order of 1 as well as $\psi^+(\mathbf{x}')$, the denominator $|\mathbf{x}-\mathbf{x}'|$ is of the order of a. Thus, the order of the value of the integral term in (2) in this case is $\frac{ma^2|V|}{\hbar^2}$ and the criterion of the Born approximation validity is $\frac{ma^2|V|}{\hbar^2} \ll 1$ or $|V| \ll \frac{\hbar^2}{ma^2}$. The value $\frac{\hbar^2}{ma^2}$ has meaning of kinetic energy for a particle confined in a volume with the linear size a.

Let consider now the short-wave limit $ka \gg 1$. Then the exponent in equation (2) changes rapidly in the volume with the linear size a and the previous estimate becomes invalid. To improve it let substitute $e^{i\mathbf{k}_i\mathbf{x}'}$ instead of $\psi^+(\mathbf{x}')$ and multiply and divide the integral by the factor $e^{i\mathbf{k}_i\mathbf{x}}$. The result reads:

$$\int \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} V(\mathbf{x}') \psi^+(\mathbf{x}') d^3x' \approx e^{i\mathbf{k}_i\mathbf{x}} \int \frac{e^{ik|\mathbf{x}-\mathbf{x}'|-i\mathbf{k}_i(\mathbf{x}-\mathbf{x}')}}{|\mathbf{x}-\mathbf{x}'|} V(\mathbf{x}') d^3x' \quad ((3))$$

The exponent in the integral in eq. (3) can be rewritten as $e^{ik|\mathbf{x}-\mathbf{x}'|(1-\cos\theta)}$, where θ is the angle between $\mathbf{k}_i$ and $\mathbf{x}-\mathbf{x}'$. Since ka is very large and $|\mathbf{x}-\mathbf{x}'|$ has the order of magnitude a, the oscillating exponent cancel the integrand everywhere beyond a region in which the argument of the exponent $k|\mathbf{x}-\mathbf{x}'|(1-\cos\theta)$ is less or equal to π, i.e. at $\theta \lesssim 1/\sqrt{ka}$. The integration over θ gives the factor $\sim 1/(ka)$, the rest of the integral contributes a factor $\sim |V| a^2$. Thus, the Born approximation works if $\frac{m|V|a}{\hbar^2 k} = \frac{|V|a}{\hbar v} \ll 1$, where $v = \frac{\hbar k}{m}$ is the velocity.