Weakly interacting Bose-gas

Lecture note

Let start with the Hamiltonian in momentum representation:

$$H=\sum_{\mathbf{p}}\frac{\mathbf{p}^2}{2m}a_{\mathbf{p}}^{\dagger}a_{\mathbf{p}}+\frac{g}{2V}\sum_{\mathbf{p},\mathbf{p}',\mathbf{q}}a_{\mathbf{p}}^{\dagger}a_{\mathbf{p}'}^{\dagger}a_{\mathbf{p}'+\mathbf{q}}a_{\mathbf{p}-\mathbf{q}} \tag{1}$$

When writing this formula we should remember that the second sum originates from a slightly nonlocal in space interaction

$$H_{int}=\frac{1}{2}\int g\left(\mathbf{x}-\mathbf{x}'\right)\psi^{\dagger}\left(\mathbf{x}\right)\psi^{\dagger}\left(\mathbf{x}'\right)\psi\left(\mathbf{x}'\right)\psi\left(\mathbf{x}\right)d^3x d^3x' \tag{2}$$

According to the Bogoliubov's idea, the average number of particle in the ground state N_0 is close to the total number of particles in the gas N and is very large ($\sim 10^{23}$ in liquid ^{4}He and $\sim 10^5$ in cooled gases of alkali atoms). Therefore, it is possible to neglect non-commutativity of the operators a_0 and $a_0^{\dagger}$ and substitute each of them by the number $\sqrt{N_0}$. In this approximation the Hamiltonian (1) is reduced to a quadratic one in terms of operators $a_{\mathbf{p}}$ and $a_{\mathbf{p}}^{\dagger}$ [1]:

$$H=\sum_{\mathbf{p}}\frac{\mathbf{p}^2}{2m}a_{\mathbf{p}}^{\dagger}a_{\mathbf{p}}+\frac{gN}{V}\sum_{\mathbf{p}}a_{\mathbf{p}}^{\dagger}a_{\mathbf{p}}+\frac{gN}{2V}\sum_{\mathbf{p}}\left(a_{\mathbf{p}}^{\dagger}a_{-\mathbf{p}}^{\dagger}+a_{\mathbf{p}}a_{-\mathbf{p}}\right)+\frac{gN^2}{2V} \tag{3}$$

This operator can be diagonalized by the Bogolyubov's transformation:

$$a_{\mathbf{p}}=u_{\mathbf{p}}\alpha_{\mathbf{p}}-v_{\mathbf{p}}\alpha_{-\mathbf{p}}^{\dagger};\ a_{-\mathbf{p}}^{\dagger}=-v_{\mathbf{p}}\alpha_{\mathbf{p}}+u_{\mathbf{p}}\alpha_{-\mathbf{p}}^{\dagger} \tag{4}$$

with real coefficients $u_{\mathbf{p}}$ and $v_{\mathbf{p}}$. It is useful to write down the transformation for Hermitian conjugated operators:

$$a_{\mathbf{p}}^{\dagger}=u_{\mathbf{p}}\alpha_{\mathbf{p}}^{\dagger}-v_{\mathbf{p}}\alpha_{-\mathbf{p}};\ a_{-\mathbf{p}}=-v_{\mathbf{p}}\alpha_{\mathbf{p}}^{\dagger}+u_{\mathbf{p}}\alpha_{-\mathbf{p}} \tag{5}$$

From requirement of canonical commutation relations for both sets of operators $a_{\mathbf{p}}$, $a_{\mathbf{p}}^{\dagger}$ and $\alpha_{\mathbf{p}}$, $\alpha_{\mathbf{p}}^{\dagger}$ we find the constraint:

$$u_{\mathbf{p}}^2-v_{\mathbf{p}}^2=1 \tag{6}$$

The Hamiltonian (3) decays into a sum of independent Hamiltonians for particles with momenta $\mathbf{p}$ and $-\mathbf{p}$. Let consider one of them:

$$H_{\mathbf{p}}=\xi_{\mathbf{p}}\left(a_{\mathbf{p}}^{\dagger}a_{\mathbf{p}}+a_{-\mathbf{p}}^{\dagger}a_{-\mathbf{p}}\right)+gn_0\left(a_{\mathbf{p}}^{\dagger}a_{-\mathbf{p}}^{\dagger}+a_{\mathbf{p}}a_{-\mathbf{p}}\right), \tag{7}$$

[1] In the derivation of Eq. (3) we used the fact that the number of the particles in the condensate N_0 is much larger than the number of the over-condensate particles $\sum_{\mathbf{p}}a_{\mathbf{p}}^{\top}a_{\mathbf{p}}$ and that $N_0+\sum_{\mathbf{p}}a_{\mathbf{p}}^{\top}a_{\mathbf{p}}=N$.

where $n_0 = \frac{N_0}{V}$ is the condensate density and $\xi_{\mathbf{p}} = \frac{\mathbf{p}^2}{2m} + gn_0$. Plugging the transformation formulae (4,5) into equation (7) and employing the permutation relations, we find:

$$\begin{aligned} H_{\mathbf{p}} &= \left(\alpha_{\mathbf{p}}^{\dagger}\alpha_{\mathbf{p}} + a_{-\mathbf{p}}^{\dagger}\alpha_{-\mathbf{p}}\right)\left[\xi_{\mathbf{p}}\left(u_{\mathbf{p}}^2 + v_{\mathbf{p}}^2\right) - gn_0 2u_{\mathbf{p}}v_{\mathbf{p}}\right] + \\ &\quad \left(\alpha_{\mathbf{p}}^{\dagger}\alpha_{-\mathbf{p}}^{\dagger} + \alpha_{\mathbf{p}}\alpha_{-\mathbf{p}}\right)\left[-\xi_{\mathbf{p}}2u_{\mathbf{p}}v_{\mathbf{p}} + gn_0\left(u_{\mathbf{p}}^2 + v_{\mathbf{p}}^2\right)\right] + \\ &\quad 2\xi_{\mathbf{p}}v_{\mathbf{p}}^2 - gn_0 2u_{\mathbf{p}}v_{\mathbf{p}} \end{aligned} \tag{8}$$

The Hamiltonian $H_{\mathbf{p}}$ becomes diagonal in the set of states with definite values of occupation numbers $\tilde{n}_{\mathbf{p}} = \alpha_{\mathbf{p}}^{\dagger}\alpha_{\mathbf{p}}$ if the term in the second line of equation (8) vanishes:

$$-\xi_{\mathbf{p}}2u_{\mathbf{p}}v_{\mathbf{p}} + gn_0\left(u_{\mathbf{p}}^2 + v_{\mathbf{p}}^2\right) = 0 \tag{9}$$

Equation (9) together with equation (6) uniquely determines the coefficients of the Bogoliubov transformation:

$$u_{\mathbf{p}}^2 = \frac{1}{2}\left(\frac{\xi_{\mathbf{p}}}{\varepsilon_{\mathbf{p}}} + 1\right);\ v_{\mathbf{p}}^2 = \frac{1}{2}\left(\frac{\xi_{\mathbf{p}}}{\varepsilon_{\mathbf{p}}} - 1\right), \tag{10}$$

where

$$\varepsilon_{\mathbf{p}} = \sqrt{\xi_{\mathbf{p}}^2 - (gn)^2} = \sqrt{\left(\frac{\mathbf{p}^2}{2m} + gn\right)^2 - (gn)^2} \tag{11}$$

Indeed, dividing equation (9) by $2u_{\mathbf{p}}v_{\mathbf{p}}$ one obtains for the ratio $z_{\mathbf{p}} = \frac{v_{\mathbf{p}}}{u_{\mathbf{p}}}$ a quadratic equation : $z_{\mathbf{p}}^2 - 2\frac{\xi_{\mathbf{p}}}{gn_0}z_{\mathbf{p}} + 1 = 0$. Its larger root is $z_{\mathbf{p}} = \frac{\xi_{\mathbf{p}} + \varepsilon_{\mathbf{p}}}{gn_0}$. Substituting $v_{\mathbf{p}} = z_{\mathbf{p}}u_{\mathbf{p}}$ into equation (6) one arrives at equations (10).

The transformed Hamiltonian $H_{\mathbf{p}}$ reads:

$$H_{\mathbf{p}} = \varepsilon_{\mathbf{p}}\alpha_{\mathbf{p}}^{\dagger}\alpha_{\mathbf{p}} + 2\xi_{\mathbf{p}}v_{\mathbf{p}}^2 - gn2u_{\mathbf{p}}v_{\mathbf{p}} \tag{12}$$

The first term in the right-hand side of equation (12) describes the excitations with the spectrum $\varepsilon_{\mathbf{p}}$, the last two terms contribute to the ground state energy. The excitation energy can be rewritten as follows:

$$\varepsilon_{\mathbf{p}} = p\sqrt{\frac{gn}{m} + \frac{p^2}{4m^2}} \tag{13}$$

This equation shows that at $p \ll \sqrt{2mgn}$ the spectrum is linear $\varepsilon_{\mathbf{p}} \approx sp$ where $s = \sqrt{\frac{gn}{m}}$ is the sound velocity. At large $p \gg \sqrt{2mgn}$ the spectrum coincides with the spectrum of free particle $\varepsilon_{\mathbf{p}} \approx \frac{p^2}{2m}$. The interaction is negligible in this limit.

The energy of the ground state reads

$$E_0 = \frac{gN^2}{2V} + \sum_{\mathbf{p}}\left(2\xi_{\mathbf{p}}v_{\mathbf{p}}^2 - gn2u_{\mathbf{p}}v_{\mathbf{p}}\right) = \frac{gN^2}{2V} + \sum_{\mathbf{p}}\left(\varepsilon_{\mathbf{p}} - \xi_{\mathbf{p}}\right) \tag{14}$$

The sum in (14) is proportional to $g^{5/2}n_0^{3/2}m^{3/2}N/\left(2\pi\hbar\right)^3$ and is much less than the first term if $g \ll \frac{\hbar^2}{n_0^{1/3}m}$[2]. Neglecting it , we obtain an important result:

$$E_0 \approx \frac{gN^2}{2V} \tag{15}$$

The derivative of this energy over N is the chemical potential, i.e. the energy necessary to add 1 particle to the system:

$$\mu = gn \tag{16}$$

Since the new particle goes with overwhelming probability to the condensate, the same value (16) can be treated as interaction energy per one particle in condensate. Differentiating energy (15) by volume, one finds the pressure:

$$p = -\frac{\partial E_0}{\partial V} = \frac{gn^2}{2} \tag{17}$$

According to the Laplace formula, the sound velocity s can be found from the derivative of the pressure over mass density $\rho = mn$:

$$s = \sqrt{\frac{\partial p}{\partial \rho}} = \sqrt{gn/m} \tag{18}$$

As it was shown by Landau, this type of the excitation spectrum leads to superfluidity, i.e. to the absence of dissipation. Let consider a heavy ball of the mass M moving in the Bose-gas with the velocity $\mathbf{v}$. It can be deccelerated only by exciting of the gas. The conservation of the momentum and energy reads:

$$E_{\mathbf{p}} = E_{\mathbf{p}-\mathbf{q}} + \varepsilon_{\mathbf{q}}, \tag{20}$$

where $\mathbf{p}$ $=M\mathbf{v}$ and $E_{\mathbf{p}} = \frac{\mathbf{p}^2}{2M}$. Since the change of energy of a heavy ball is very small, the difference of energy can be expressed as follows: $E_{\mathbf{p}} - E_{\mathbf{p}-\mathbf{q}} \approx \frac{\partial E}{\partial \mathbf{p}}\mathbf{q} = \mathbf{vq}$. Thus, the conservation law (20) reads:

$$\mathbf{vq} = \sqrt{s^2q^2 + \frac{q^4}{4m^2}} \tag{20}$$

The left-hand side of this equation is maximal when $\mathbf{v}$ and $\mathbf{q}$ are parallel. In this case the l.-h.side of Eq. (20)is equal to vq. It is definitely smaller than the right-hand side of equation (20) as long as $v < s$. Thus, the dissipation is possible only if $v > s$.

[2] The sum in (14) is divergent. To obtain a physically reasonable result one needs to express the interaction constant in terms of the s-scattering amplitude to the second order of perturbation theory. See the details in the book Landau and Lifshitz, Statistical Physics, part 2, p. 99-102, Pergamon, 1991.