The motion of a quantum particle due to a constant force

Lecture Notes

The potential of the constant force F reads $V = -Fx$. The stationary state is determined by the Schrödinger equation:

$$\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + (E + Fx)\,\psi = 0 \tag{1}$$

At positive F the classically permitted region extends to the right from the turning point $x_0 = -\frac{E}{F}$. By a change of variables:

$$x + \frac{E}{F} = \left(\frac{\hbar^2}{2mF}\right)^{1/3}\xi \tag{2}$$

the Schrödinger equation is reduced to a standard dimensionless form:

$$\frac{d^2\psi}{d\xi^2} + \xi\psi = 0 \tag{3}$$

Its solution will be sought by a complex modification of the Laplace transformation as a contour integral:

$$\psi(\xi) = \int\limits_{\Gamma} e^{\lambda\xi}\phi(\lambda)\,d\lambda \tag{4}$$

The only reuirements to the contour Γ in the complex plane λ are that it can not be deformed to a point and that the value of the Laplace transform $\phi(\lambda)$ at the contour ends must be zero. Since differentiation over ξ commutes with the integration over λ, the equation (3) can be rewritten as follows:

$$\int\limits_{\Gamma} e^{\lambda\xi}\left(\lambda^2 + \xi\right)\phi(\lambda)\,d\lambda = 0 \tag{5}$$

The second term in this equation can be represented as follows:

$$\xi\int\limits_{\Gamma} e^{\lambda\xi}\phi(\lambda)\,d\lambda = -\int\limits_{\Gamma} e^{\lambda\xi}\frac{d\phi(\lambda)}{d\lambda}d\lambda \tag{6}$$

Indeed, after the integration by parts of the integral in the r.-h. s. of equation (6), keeping in mind that $\phi(\lambda)$ is zero at the contour ends, we obtain the expression in the left-hand side of the same equation. Thus, equation (3) will be satisfied if $\phi(\lambda)$ satisfies the first order differential equation:

$$\frac{d\phi(\lambda)}{d\lambda} - \lambda^2\phi(\lambda) = 0 \tag{7}$$

The general solution of this equation reads:

$$\phi(\lambda) = C \exp\left(\frac{\lambda^3}{3}\right) \tag{8}$$

where C is an arbitrasry constant, and, according to (4), the general solution of Airy equation reads:

$$\psi(\xi) = C \int\limits_{\Gamma} e^{\frac{\lambda^3}{3}+\lambda\xi} d\lambda \tag{9}$$

It seems that it is determined by only one constant C in contradiction with the general fact that there are two independent solutions of the second order differential equation. However, this contradiction is fictitious since we still did not choose the contour Γ. We will see that there are only two possible contours which can not be reduced each to other. The function $\phi(\lambda)$ defined by equation (8) does not turn into zero at any finite λ. Therefore, the ends of the contour Γ must be at infinity. The exponent in equation (8) decreases at infinity if $\mathrm{Re}\,\lambda^3 < 0$. Using the polar representation of the complex number $\lambda = |\lambda|\, e^{i\varphi}$, one can see that $\mathrm{Re}\,\lambda^3$ is negative if $\cos 3\varphi < 0$. It happens if 3φ is confined to one of 3 intervals: $\frac{\pi}{2} < 3\varphi < \frac{3\pi}{2}$; $-\frac{3\pi}{2} < 3\varphi < -\frac{\pi}{2}$, or $\frac{5\pi}{2} < 3\varphi < \frac{7\pi}{2}$. This means, that φ is confined in the following 3 sectors: 1) $\frac{\pi}{6} < \varphi < \frac{\pi}{2}$; 2) $-\frac{\pi}{2} < \varphi < -\frac{\pi}{6}$, or 3) $\frac{5\pi}{6} < \varphi < \frac{7\pi}{6}$. Each of this sectors has angular width $\frac{\pi}{3}$. They are shown in Fig. 1. Possible contours must start in one of the three sectors and end in another. The contours which start and end in the same sector can be deformed into a point. It means that the integral along such a contour is zero. There are only two independent solutions. One of them corresponds to a contour which starts in the sector 1) and ends in the sector 2); the contour corresponding to the second solution starts in 1) and ends in 3). The remaining choice of the contour going from 2) to 3) gives the solution, which is the difference of the second and the first solutions as it is clearly seen on Fig. 1.

We are looking for a solution which exponentially decreases in classically forbidden region $\xi < 0$. Therefore, it is described by a real wave function as any state with complete reflection. The contour Γ which corresponds to the real solution must be deformable into a contour symmetric with respect to real axis λ. Then upper and lower parts of this contour pass through complex conjugated values of λ at which the inegrand and the differentials $d\lambda$ accept complex conjugated values at real ξ. Only the contour passing through the regions 1 and 2 has this property. The solution (9) with the constant $C = (2\sqrt{\pi})^{-1}$ is called the Airy function and denoted $\mathrm{Ai}\,(-\xi)$. The contour Γ can be deformed to the imaginary axis. Then by the change of variable $\lambda = iu$ the integral (9) can be transformed to the following form:

$$\mathrm{Ai}\,(-\xi) = \frac{1}{\sqrt{\pi}} \int\limits_0^\infty \sin\left(\frac{u^3}{3} - \xi u\right) du \tag{10}$$

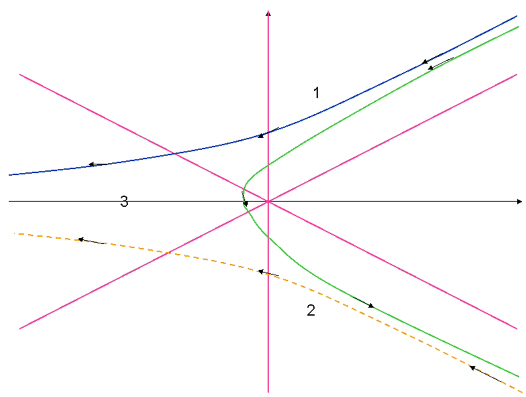

Fig. 1. Complex plane λ and contours Γ.
The numbers 1,2,3 indicate the three sectors in which the ends of the contours are confined. Two independent contours are marked by green and blue colurs. The third contour (orange) can be treated as passing first the green contour and then the blue one in the opposide direction.

To find the asymtotic behaviour of the Airy function we apply the saddle point method (see Lecture Note) to the integral (9). The argument of the exponent is $f(\lambda) = \frac{\lambda^3}{3} + \lambda\xi$. The saddle point is determined by equation:

$$f'(\lambda) = \lambda_s^2 + \xi = 0 \tag{11}$$

There are two saddle points: $\lambda_s = \pm\sqrt{-\xi}$. At $\xi < 0$ they are located on real axis; at positive ξ the two saddle points are located on the imaginary axis. The direction of the steepest descent is determined the second derivative $f"(\lambda_s) = 2\lambda_s = \pm 2\sqrt{-\xi}$. According to the general rule, it is tilted to the real axis by the angle $\alpha = \pm\frac{\pi}{2} - \frac{1}{2}\arg\lambda_s$. For negative ξ one of saddle point is positive and its argument is equal to zero, whereas another saddle point is negative and its argument is equal to π. Thus, the steepest descent line is perpendicular to the real axis for $\lambda_s > 0$ and it is directed along real axis for $\lambda_s < 0$. The contur Γ can be easily deformed in such a way that it remains symmetric with respect to real axis and passes through the right saddle point in the direction of the steepest decent, but it is impossible for the left saddle point (Fig. 2). Thus, at negative ξ the contour passes only through one saddle point $\lambda_s = +\sqrt{-\xi}$. In this point $f\left(\sqrt{-\xi}\right) = -\frac{2}{3}(-\xi)^{3/2}$. It gives the following answer for the asymptotic of Airy function at large negative ξ:

$$\mathrm{Ai}(-\xi) \simeq \frac{1}{2(-\xi)^{1/4}} \exp\left[-\frac{2}{3}(-\xi)^{3/2}\right]; \quad \xi < 0, |\xi| \gg 1 \tag{12}$$

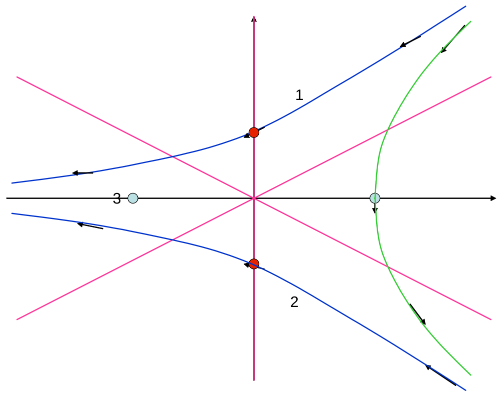

Fig. 2. Saddle points and paths of steepest descent. Green points correspond to negative ξ, red points correspond to positive ξ.

At large positive ξ the two saddle points are purely imaginary, their arguments are $\pm\frac{\pi}{2}$ and the steepest descent directions are tilted by the angle $\pm\pi/4$ to the real axis at the upper and lower saddle point, respectively (Fig. 2). Thus, both turning points contribute to the asymptotic of the Airy function at large positive ξ. The final result reads:

$$\mathrm{Ai}\left(-\xi\right) \simeq \frac{1}{\left(\xi\right)^{1/4}} \cos\left(\frac{2}{3}\left(\xi\right)^{3/2} - \frac{\pi}{4}\right) \qquad (13)$$