Chapter 13

Last Time

Chapter 11
• Pre-lecture
• “Tipping box” example

Chapter 13
• Gravitational force
• Generalized expression for gravitational potential energy

Today

Chapter 13
• Satellite motion and Circular Orbits
• Kepler’s Laws
• Chapter 13 examples

Don’t forget: Exam 3 one week from today
Class for the Wednesday before Thanksgiving is canceled and the time lost is being made up during your exam 3.
Example Conservation of energy

Consider two identical objects released from rest high above the surface of the earth (neglect air resistance). In Case 1 we release an object from a height above the surface of the earth equal to 1 earth radius, and in Case 2 we release an object from a height above the surface of the earth equal to 2 earth radii.

Compare the kinetic energy of the two objects just before they hit the surface of the earth.

\[ K_i + U_i = K_f + U_f \]

Case 1

\[ 0 + \frac{-GMMe}{(2Re)} = \frac{1}{2}mv_i^2 - \frac{6MM_e}{(Re)} \]

\[ v_i^2 = 2\left(\frac{GMMe}{2Re} + \frac{GMMe}{Re}\right) \]

\[ v_i = -GMMe \left(\frac{-1}{Re} + \frac{2}{2Re}\right)^{1/2} = \sqrt{\frac{GMMe}{Re}} \]

Case 2

\[ 0 + \frac{-GMMe}{3Re} = \frac{1}{2}mv_2^2 - \frac{GMMe}{Re} \]

\[ v_2^2 = \frac{-GMMe}{3Re} + \frac{GMMe}{Re} \]

\[ v_2 = \sqrt{\frac{2GMMe}{3Re}} \]

\[ K_i = \frac{1}{2}m \left(\frac{GMMe}{Re}\right) \]

\[ K_f + U_f = K_i + U_i + W_{\text{other}} \]

\[ F_{\text{grav}} = G\frac{M_1M_2}{R_{12}^2} \quad U_{\text{grav}} = -G\frac{M_1M_2}{R_{12}} \quad T = \frac{2\pi a^{3/2}}{\sqrt{GM}} \]
Satellite motion

- Orbits are in the family of conic sections: Parabola, circle, ellipse, and hyperbola

Circular orbits:
- The simplest orbits are circular
- Using Newton’s 2\textsuperscript{nd} and his Law of Gravity we can solve for the motion of the satellite in circular orbit:

\[
\sum \vec{F} = m\vec{a}_{rad} \Rightarrow \frac{GM_E}{r^2} = \frac{mv^2}{r}
\]

Solving for velocity:

\[v = \sqrt{\frac{GM_E}{r}}
\]

(for a given radius the velocity is determined)
Circular satellite motion

Given the equations of circular motion for a satellite:

\[ v_{\text{orbit}} = \sqrt{\frac{GM_E}{r_{\text{orbit}}}} \] and

\[ T_{\text{orbit}} = 2\pi \sqrt{\frac{r_{\text{orbit}}^3}{GM_E}} \]

\[ v = \frac{2\pi r}{T} \]

\[ T = \frac{2\pi r}{v} = 2\pi \sqrt{\frac{r_{\text{orbit}}}{6ME}} = 2\pi \sqrt{\frac{r^3}{6ME}} \]

- If \( v < v_{\text{orbit}} \) then the particle will fall back down
- If \( v > v_{\text{orbit}} \) then the particle will either:
  - be in an elliptical closed orbit \( \Rightarrow v < \sqrt{\frac{GM_E}{R}} = v_{\text{esc}} \)
  - escape the gravitational field of the object \( \Rightarrow v > v_{\text{esc}} \)

- It will **only** be in a circular orbit if \( \Rightarrow v = v_{\text{orbit}} \)
**Example Circular orbit**

On July 15, 2004, NASA launched the Aura spacecraft to study the earth’s atmosphere. This satellite was placed in an orbit 705 km above the earth’s surface, in a circular orbit. Assume it’s mass is 1000 kg.

a) How many hours does it take to make one orbit?

b) How fast is it moving?

c) How much work had to be done to place this satellite in orbit?

\[
T = \frac{2\pi \Gamma}{V} \quad \frac{GmM_e}{r^2} = \frac{mv^2}{r} \quad V = \sqrt{\frac{GmM_e}{r}}
\]

\[
V = \sqrt{\frac{6.67 \times 10^{-11} (5.97 \times 10^{24})}{6.38 \times 10^6 + 7.05 \times 10^3}} = 7497 \text{ m/s}
\]

\[
T = \frac{2\pi (6.38 \times 10^6 + 7.05 \times 10^3)}{(7497 \text{ m/s})} \times \frac{1 \text{ hr}}{3600 \text{ s}} = 1.65 \text{ hrs}
\]

\[
W = -\int F_e \cdot dr = -\Delta U = U_1 - U_2 = \left( -\frac{GmM_e}{r} + \frac{GmM_e}{r + h} \right)
\]

\[
W = 2.55 \times 10^{12} \text{ J}
\]

\[
M_e = 5.97 \times 10^{24} \text{ kg} \\
R_e = 6.38 \times 10^6 \text{ m} \\
G = 6.67 \times 10^{-11} \text{ Nm/kg}^2
\]
Clicker Question

Suppose the Sun were to shrink to half of its present radius while maintaining the same mass. What effect would this have on the Earth’s orbit?

A. The size of the orbit would decrease and the orbital period would decrease.

B. The size of the orbit would increase and the orbital period would increase.

C. The size of the orbit and the orbital period would remain unchanged.

D. none of these

\[
\sum \vec{F} = m\vec{a}, \quad \vec{F}_B \text{ on } A = -\vec{F}_A \text{ on } B \quad |\vec{a}_{\text{rad}}| = \frac{v^2}{R} \quad T = \frac{2\pi R}{v}
\]

\[
F_{\text{grav}} = G\frac{M_1 M_2}{R_{12}^2} \quad U_{\text{grav}} = -G\frac{M_1 M_2}{R_{12}} \quad T = \frac{2\pi a^{3/2}}{\sqrt{GM}}
\]
Kepler’s laws and planetary motion

**Kepler’s Laws**

1) Each planet moves in an elliptical orbit, with the sun at one focus of the ellipse.

2) A line from the sun to a given planet sweeps out equal areas in equal times.

3) The period of the planets are proportional to \( (2a)^{3/2} \) where \( a \) is the length of the major axis of their orbit.
Kepler’s 1\textsuperscript{st} Law

Ellipses

• An ellipse is the intersection of a cone with a plane
• A circle is the special case in which $e = 0$

$$\frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{a^2(1 - e^2)} = 1$$

• Where $e$ is the eccentricity and $a$ is the major axis

$$r = \frac{ab}{\sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta}} \quad \text{or} \quad r = \frac{a(1 - e^2)}{1 \pm e \cos \theta}$$

if $a = b$ then $r = a$

if $e = 0$ then $r = a$

Newton showed that the only possible closed orbits for an object acting under an attractive force (proportional to $1/r^2$) are a circle and an ellipse

He also showed that open orbits were parabolas or hyperbolas.
Kepler’s 2nd Law

- Area of the triangle swept out in time $dt$ is:
  
  $$dA = \frac{1}{2} \cdot (\text{base})(\text{height}) = \frac{1}{2} (r \, d\theta)(r)$$

  $$dA = \frac{1}{2} r^2 \, d\theta$$

  $$\Rightarrow \frac{dA}{dt} = \frac{r^2}{2} \frac{d\theta}{dt}$$

- Since $v_\perp = v \sin \phi$ and the distance traveled in time $dt$ is $r \, d\theta$, this means that

  $$v_\perp = r \frac{d\theta}{dt} \Leftrightarrow \frac{d\theta}{dt} = \frac{v}{r} \sin \phi$$

  $$\Rightarrow \frac{dA}{dt} = \frac{1}{2} rv \sin \phi$$

  $$\Rightarrow rv \sin \phi = \vec{r} \times \vec{v}$$

  $$\vec{L} = \vec{r} \times \vec{p} = m(\vec{r} \times \vec{v})$$

- If momentum is conserved then $\frac{dA}{dt}$ is constant

  $$\frac{dA}{dt} = \frac{\vec{L}}{2m}$$
Kepler’s 3\textsuperscript{rd} Law

- Newton showed that the result for the circular orbit of a satellite holds true for ellipses in general, using the major axis, a, instead of the radius, r.

\begin{equation}
T_{\text{orbit}} = 2\pi \sqrt{\frac{a^3}{GM_E}} \propto a^{\frac{3}{2}}
\end{equation}

\text{(the proof is beyond the scope of this course but I promise it works)}

- Note that the result does not depend on e.

That is \textbf{NOT} acceleration!!!
Kepler's second law states that a line from the sun to a planet sweeps out equal areas in equal times.

The aphelion is the point in the orbit where the planet is the furthest distance from the sun, The perihelion is the point at which the planet is closest to the sun.

At which point must the speed of the planet be greater?

a) At the perihelion
b) At the aphelion
c) speed is constant
An apple hanging from a tree is subject to a gravitational pull from the Earth. The apple also exerts a gravitational force on the Earth. The force exerted by the apple on the Earth is

a) equal in magnitude to the force exerted by the Earth on the apple.
b) greater in magnitude than the force exerted by the Earth on the apple.
c) smaller in magnitude than the force exerted by the Earth on the apple.
A planet \((P)\) is moving around the Sun \((S)\) in an elliptical orbit. As the planet moves from aphelion to perihelion, the Sun’s gravitational force

- A. does positive work on the planet.
- B. does negative work on the planet.
- C. does positive work on the planet during part of the motion and negative work during the other part.
- D. does zero work on the planet at all points between aphelion and perihelion.
Example eoc 13.23

Deimos, a moon of Mars, is about 12 km in diameter with mass 2.00x10^{15} kg.

a) With what speed would you have to throw a baseball so that it would go into a circular orbit 1.00m above the surface?

\[
\text{circ.} \quad G \frac{mM_D}{r^2} = \frac{mv^2}{r}
\]

\[
v = \sqrt{\frac{GM_D}{r}} = \sqrt{\frac{6.67 \times 10^{-11} \left(2.00 \times 10^{15}\right)^2}{(6 \times 10^3 + 1)^2}} = 4.7 \text{ m/s}
\]

\[
T = \frac{2\pi r}{v} = \frac{2\pi (600 \text{ m})}{4.7 \text{ m/s}} = 8020 \text{ s} \times \frac{1}{3600 \text{s}} = 2.23 \text{hrs}
\]

b) How long after throwing the ball would it return to you?

\[
G = 6.67 \times 10^{-11} \text{Nm}^2/\text{kg}^2
\]

\[
|\vec{a}_{\text{rad}}| = \frac{v^2}{R} \quad T = \frac{2\pi R}{v}
\]

\[
F_{\text{grav}} = G \frac{M_1 M_2}{R_{12}^2} \quad U_{\text{grav}} = -G \frac{M_1 M_2}{R_{12}} \quad T = \frac{2\pi a^{3/2}}{\sqrt{GM}}
\]
Clicker Question

The planet Saturn has 100 times the mass of the Earth and is 10 times more distant from the Sun than the Earth is.

Compared to the Earth’s acceleration as it orbits the Sun, the acceleration of Saturn as it orbits the Sun is

\[ F = m a = m \left( \frac{G M}{r^2} \right) \]

A. 100 times greater.

B. 10 times greater.

C. the same.

D. 1/10 as great.

E. 1/100 as great.

\[ G = 6.67 \times 10^{-11} \text{Nm}^2/\text{kg}^2 \]

\[ |\ddot{r}_{\text{rad}}| = \frac{v^2}{R} \quad T = \frac{2\pi R}{v} \]

\[ F_{\text{grav}} = G \frac{M_1 M_2}{R_{12}^2} \quad U_{\text{grav}} = -G \frac{M_1 M_2}{R_{12}} \quad T = \frac{2\pi a^{3/2}}{\sqrt{GM}} \]
Many satellites move in a circle in the earth’s equatorial plane. They are at a height above the surface that they always remain above the same point.

a) Find the altitude of these satellites.

b) Why can’t signals from these satellites reach receivers on the earth’s surface above the 81.3°N latitude line?

\[ T = 24 \text{hrs} \times \frac{3600 \text{s}}{1 \text{hr}} = 8.64 \times 10^4 \text{s} \]

\[ T = \frac{2\pi a^{3/2}}{\sqrt{GM_e}} \quad \Rightarrow \quad T \sqrt{GM_e} = a^{3/2} \]

\[ \left( a^{3/2} \right)^{2/3} = \left( \frac{T \sqrt{GM_e}}{2\pi} \right)^{2/3} = R_e + h \]

\[ h = \left( \frac{T \sqrt{GM_e}}{2\pi} \right)^{2/3} - R_e = \left( \frac{8.64 \times 10^4}{2\pi \times \sqrt{6.67 \times 10^{-11} \times (6.97 \times 10^{24})}} \right) - (6.38 \times 10^6) \]

\[ h = 3.58 \times 10^7 \text{m} \]
Example

A comet of mass $1.00 \times 10^9 \text{ kg}$ is observed at a distance from the sun of $8.00 \times 10^{11} \text{ m}$ at a speed of $17000 \text{ m/s}$. Assuming no forces on it other than the sun's gravity, how fast will it be going when it is a distance of $2.25 \times 10^{11} \text{ m}$ from the sun?

$$E_{\text{tot}} = \frac{1}{2} m v^2 - \frac{G m M_o}{R}$$

$$\frac{1}{2} m v_i^2 - \frac{G m M_o}{r_1} = \frac{1}{2} m v_f^2 - \frac{G m M_o}{r_2}$$

$$v_f^2 = v_i^2 + 2 G M_o \left( \frac{1}{r_2} - \frac{1}{r_1} \right)$$

$$v_f = 3.37 \times 10^4 \text{ m/s}$$