Chapter 9 - Introduction (circular motion review)

Rotational motion of rigid bodies

- Up to now we have only talked about point-like particles, obviously they aren’t actually points and have a finite size.
- The motion of a rigid body (finite sized object) is described by two things: the translation of its center of mass and the rotation about that center of mass.
- Focus for now just on the rotation, we will get back to the translation later.
- When a body is rotating all points on that body are moving in circles, they all rotate about a line called the axis of rotation.
Rotational coordinate: the radian

- The natural unit for angles is the radian, one radian is the angle such that the arc-length \( s \) is equal to the radius \( R \).

\[
360° = \frac{2\pi R}{R} = 2\pi \text{ rad}
\]

- The value of an angle \( \theta \) is equal to the ratio of the arc-length and the radius

\[
\theta \equiv \frac{s}{R} \quad (\text{ratio of two lengths, so no units})
\]

- The direction of circular motion follows the convention:
  - Counter-clockwise is positive
  - Clockwise is negative
Polar coordinates review

Cartesian
\[ \vec{r} = r_x \hat{i} + r_y \hat{j} \]

Polar
\[ \vec{r} = r \hat{r} \]

Angular displacement is \( \Delta \theta = \theta_2 - \theta_1 \)

Define the time rate of change of \( \theta \) as \( \omega \)

\[ \omega \equiv \frac{d\theta}{dt} \]

Technically \( \omega \) is a vector that points perpendicular to the plane defined by the rotation. Practically speaking it comes down to positive rotation (RHR, thumb along the +z-axis) and negative rotation (RHR, thumb along the –z-axis)

Counterclockwise rotation positive:
\( \Delta \theta > 0 \), so
\( \omega_{av-z} = \Delta \theta / \Delta t > 0 \)

Clockwise rotation negative:
\( \Delta \theta < 0 \), so
\( \omega_{av-z} = \Delta \theta / \Delta t < 0 \)

Axis of rotation (z-axis) passes through origin and points out of page.
Angular acceleration

- Time rate of change of the angular velocity $\omega$: $\alpha \equiv \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$ (units of rad/s$^2$)

- If we have constant $\alpha$ then we can do the same derivations we did for constant linear acceleration to get the equations of motion for constant angular acceleration:

  Constant linear acceleration, $\alpha$

  \[
  x = x_0 + v_0 t + \frac{1}{2} at^2 \\
  v = v_0 + at \\
  v^2 = v_0^2 + 2a(x - x_0) \\
  x = x_0 + \frac{1}{2}(v_0 + v)t
  \]

  Constant rotational acceleration, $\alpha$

  \[
  \theta = \theta_0 + \omega_0 t + \frac{1}{2} \alpha t^2 \\
  \omega = \omega_0 + \alpha t \\
  \omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0) \\
  \theta = \theta_0 + \frac{1}{2}(\omega_0 + \omega)t
  \]
Prelecture question 1

Two bugs (Buzz and Crunchy) are sitting on a spinning disk. Crunchy is sitting at the outer edge and Buzz is sitting half way out as shown.

Which of the following statements best describes their motion?

a) Buzz and Crunchy have the same angular velocity.

b) Buzz and Crunchy are moving with the same speed.

c) Both of the above choices are true.
Checkpoint question 1

A wheel which is initially at rest starts to turn with a constant angular acceleration. After 4 seconds it has made 4 complete revolutions.

How many revolutions has it made after 8 seconds?

a) 8
b) 12
c) 16
The graph shows the angular velocity and angular acceleration versus time for a rotating body. At which of the following times is the rotation speeding up at the greatest rate?

A. $t = 1 \text{ s}$
B. $t = 2 \text{ s}$
C. $t = 3 \text{ s}$
D. $t = 4 \text{ s}$
E. $t = 5 \text{ s}$
Buzz and Crunchy are both rotating on a disk at different radii.

**Things that are the same for Buzz and Crunchy?**

- Radians rotated through - $\theta$
- Angular velocity - $\omega$
- Angular acceleration - $\alpha$

**Things that are different?**

- Total linear distance - $x$
- Linear velocity – $v$
- Linear acceleration - $a$
Example Simple rotation

A disk spins at 78 rpm and it takes 3.5s to reach this speed at constant angular acceleration.

a) What is the angular acceleration in rad/s²?

b) How many degrees did it spin through while speeding up?

\[ \alpha = \text{const.} \]

\[
\begin{align*}
78 \text{rev/min} & \times \frac{1 \text{min}}{60 \text{s}} \times \frac{2\pi \text{ rad}}{1 \text{ rev}} = 8.17 \text{ rad/s} = \omega \quad \text{at} \ t = 3.5 \text{s} \\
\omega &= \omega_0 + \alpha t \\
\alpha &= \frac{\omega}{t} = \frac{8.17 \text{ rad/s}}{3.5 \text{s}} = 2.33 \text{ rad/s}^2
\end{align*}
\]

Remember:

\[
\begin{align*}
\theta &= \theta_0 + \omega_0 t + \frac{1}{2} \alpha t^2 \\
\omega &= \omega_0 + \alpha t \\
\omega^2 &= \omega_0^2 + 2\alpha(\theta - \theta_0) \\
\theta &= \theta_0 + \frac{1}{2}(\omega_0 + \omega)t
\end{align*}
\]

\[
\begin{align*}
\theta &= \theta_0 + \omega_0 t + \frac{1}{2} \alpha t^2 \\
&= \frac{1}{2}(2.33 \text{ rad/s})(3.5 \text{s})^2 \\
&= 14.3 \text{ rad} \times \frac{360^\circ}{2\pi \text{ rad}} = 818^\circ \approx 2.3 \text{ rev}
\end{align*}
\]
Relating linear and angular kinematics

Back to Buzz and Crunchy...

We know that they have the same angular velocity, how does this relate to the linear kinematics we are used to?

Start with \( \mathbf{r} = r \hat{r} \) (this should look familiar)

Do some calculus...

\[
\mathbf{v} = \frac{dr}{dt} \hat{r} + r \frac{d\theta}{dt} \hat{\theta}
\]

For uniform circular motion \( r \) isn’t changing:

\[
\mathbf{v} = r \frac{d\theta}{dt} \hat{\theta}
\]

In order to relate linear acceleration to rotational acceleration:

Do a bit more calculus....

\[
\ddot{\mathbf{a}} = \frac{d\mathbf{v}}{dt} = -r \left( \frac{d\theta}{dt} \right)^2 \hat{r} + r \frac{d\omega}{dt} \hat{\theta}
\]

\[
\ddot{a}_{rad} = -r \omega^2 \hat{r}
\]

\[
\ddot{a}_\text{tan} = r \alpha \hat{\theta}
\]

\[
|\ddot{a}_\text{tot}| = \sqrt{\ddot{a}_{rad}^2 + \ddot{a}_\text{tan}^2}
\]

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Example Simple rotation

A flywheel with a radius of 0.300m starts from rest and accelerates with a constant angular acceleration of 0.600 rad/s². Compute the magnitude of the tangential acceleration and the radial acceleration and the resultant acceleration of a point on the rim:

a) At t=0?

b) after it has turned through 60°?

\[ a_{\tan} = r \alpha = (0.300\text{m})(0.600\text{rad/s}^2) = 0.180\text{m/s}^2 \]

\[ a_{\text{rad}} = \omega_0^2 r = 0 \]

\[ 60^\circ \times \frac{2\pi \text{rad}}{360^\circ} = 1.05 \text{ rad} \]

\[ a_{\tan} = r \alpha = 0.180\text{m/s}^2 \]

\[ a_{\text{rad}} = \omega^2 r \]

\[ \omega = \omega_0 + 2 \alpha (\theta - \theta_0) \]

\[ \omega = 2(0.600\text{rad/s})^2(1.05\text{rad}) = 3.78\text{m/s} \]

\[ a_{\text{rad}} = 2(0.600\text{rad/s})^2(1.05\text{rad})(0.300\text{m}) = 3.78\text{m/s} \]

\[ a_{\text{tot}} = \sqrt{a_{\text{rad}}^2 + a_{\tan}^2} = 4.19 \text{m/s}^2 \]

\[ \theta = \arctan \left( \frac{a_{\tan}}{a_{\text{rad}}} \right) = 25.5^\circ \]

Remember:

\[ \theta = \theta_0 + \omega_0 t + \frac{1}{2} \alpha t^2 \]

\[ \omega = \omega_0 + \alpha t \]

\[ \omega^2 = \omega_0^2 + 2 \alpha (\theta - \theta_0) \]

\[ \theta = \theta_0 + \frac{1}{2}(\omega_0 + \omega) t \]

\[ \ddot{a}_{\text{rad}} = -r \omega^2 \]

\[ \ddot{a}_{\tan} = r \alpha \]

\[ |\ddot{a}_{\text{tot}}| = \sqrt{a_{\text{rad}}^2 + a_{\tan}^2} \]
A DVD is rotating with an ever-increasing speed. How do the centripetal acceleration $a_{\text{rad}}$ and tangential acceleration $a_{\text{tan}}$ compare at points $P$ and $Q$?

A. $P$ and $Q$ have the same $a_{\text{rad}}$ and $a_{\text{tan}}$.

B. $Q$ has a greater $a_{\text{rad}}$ and a greater $a_{\text{tan}}$ than $P$.

C. $Q$ has a smaller $a_{\text{rad}}$ and a greater $a_{\text{tan}}$ than $P$.

D. $P$ and $Q$ have the same $a_{\text{rad}}$, but $Q$ has a greater $a_{\text{tan}}$ than $P$.

$\mathbf{a}_{\text{tan}} = r\alpha \hat{\Theta}$  
$\mathbf{a}_{\text{rad}} = -r\omega^2 \hat{r}$

Angular acceleration and velocity are the same for both $P$ and $Q$ so the magnitude only depends on $r$. 

Clicker Question

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Kinetic energy of rotation

The rotational analog of mass is called the rotational inertia: Just like it takes more energy to move something the more mass it has, it takes more energy to rotate something the more rotational inertia it has.

- If we think of a body being made up of a large number of particles with masses \( m_1, m_2, \) etc ...at distances \( r_1, r_2, \) ... from the axis of rotation

- The total kinetic energy of a system of particles

\[
K_{tot} = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + ... = \sum_i \frac{1}{2} m_i v_i^2
\]

- Substitute in for \( v_i : \)

\[
K = \sum_i \frac{1}{2} m_i (r_i \omega)^2
= \frac{1}{2} \omega^2 \left( \sum_i m_i r_i^2 \right)
\]
Moment of inertia

- Define moment of inertia as the mass of each particle multiplied by its distance from the axis of rotation and adding each product together:

\[ K = \frac{1}{2} \omega^2 \left( \sum_i m_i r_i^2 \right) \]

\[ I = \sum_i m_i r_i^2 \] (moment of inertia)

\[ K = \frac{1}{2} I \omega^2 \] (rotational kinetic energy of a rigid body)

- This is not a new equation; it is just the sum of the kinetic energies of individual particles that make up a rotating rigid body.
- \( \omega \) must be in rad/sec
A square shape is made from identical point particles and identical rigid, massless rods as shown. The moment of inertia for rotations about the A and B axes (dashed lines) is $I_A$ and $I_B$ respectively.

How do $I_A$ and $I_B$ compare?

a) $I_A > I_B$

b) $I_A = I_B$

c) $I_A < I_B$

Using $I = \sum_i m_i r_i^2$

$I_A = mL^2 + mL^2 + mL^2 + mL^2 = 4mL^2$

$I_B = m(2L)^2 + m(2L)^2 = 8mL^2$
A triangular shape is made from identical balls and identical rigid, massless rods as shown. The moment of inertia about the a, b, and c axes is $I_a$, $I_b$, and $I_c$ respectively.

Which of the following orderings is correct?

a) $I_a > I_b > I_c$

b) $I_a > I_c > I_b$

c) $I_b > I_a > I_c$
Clicker Question - Previously Checkpoint question 3

In both cases shown below a hula hoop with mass M and radius R is spun with the same angular velocity about a vertical axis through its center. In Case 1 the plane of the hoop is parallel to the floor and in Case 2 it is perpendicular.

In which case does the spinning hoop have the most kinetic energy?

a) Case 1  
b) Case 2  
c) Same
Moments of inertia for basic volumes

(a) Slender rod, axis through center
\[ I = \frac{1}{12} ML^2 \]

(b) Slender rod, axis through one end
\[ I = \frac{1}{3} ML^2 \]

(c) Rectangular plate, axis through center
\[ I = \frac{1}{12} M(a^2 + b^2) \]

(d) Thin rectangular plate, axis along edge
\[ I = \frac{1}{3} Ma^2 \]

(e) Hollow cylinder
\[ I = \frac{1}{2} M(R_1^2 + R_2^2) \]

(f) Solid cylinder
\[ I = \frac{1}{2} MR^2 \]

(g) Thin-walled hollow cylinder
\[ I = MR^2 \]

(h) Solid sphere
\[ I = \frac{2}{5} MR^2 \]

(i) Thin-walled hollow sphere
\[ I = \frac{2}{3} MR^2 \]
Parallel-Axis theorem

- The moment inertia about the center-of-mass axis is related to the moment of inertia about another axis.

- It is called the parallel axis theorem.
  - For any second axis of rotation parallel to the original (in this case about the center of mass) but a distance $d$ away the moment of inertia can be written as:

$$I_p = I_{CM} + Md^2$$

Simple example of this:

(A) \( I = MR^2 \)  
(B) \( I = MR^2 + M(R)^2 = 2MR^2 \)
Example Conservation of energy and rotation

You wrap a light, non-stretching cable around a solid cylinder with mass \( m_c \) and radius \( R \). The cylinder rotates with negligible friction about a stationary axis. Tie the free end of the cable onto a block of mass \( m_b \) and release the block from rest a distance \( h \) above the floor.

Find the expressions for the speed of the falling block and the angular speed of the cylinder as the block strikes the floor.

\[
E_i = K_{i,\text{trans}} + K_{i,\text{rot}} + U_i + W_{\text{grav}}
\]

\[
E_f = K_{f,\text{trans}} + K_{f,\text{rot}} + U_f
\]

\[ K_{i,\text{trans}} = \frac{1}{2} m_b v_i^2 \quad K_{f,\text{rot}} = \frac{1}{2} I \omega_f^2 \]

\[ U_i = m_b g h \]

\[ U_f = 0 \]

\[
V = R \omega
\]

\[
v = \sqrt{\frac{m_b g h}{\frac{1}{2} m_b + \frac{1}{4} m_c}}
\]

\[
\omega = \frac{V}{R} = \frac{1}{R} \left( \frac{m_b g h}{\frac{1}{2} m_b + \frac{1}{4} m_c} \right)^{1/2}
\]

Remember:

\[
I_{\text{cyl}} = \frac{1}{2} MR^2
\]

\[
K_{\text{rot}} = \frac{1}{2} I \omega^2
\]

\[
K_{\text{tot}} = K_{\text{trans}} + K_{\text{rot}}
\]
The three objects shown here all have the same mass $M$ and radius $R$. Each object is rotating about its axis of symmetry (shown in blue). All three objects have the same rotational kinetic energy. Which one is rotating fastest?

\[ I = \frac{1}{2} M(R_1^2 + R_2^2) \]
\[ I = \frac{1}{2} MR^2 \]
\[ I = MR^2 \]

\[ R_2 = R \]
\[ K = \frac{1}{2} I \omega^2 \]

A. thin-walled hollow cylinder

B. solid sphere

C. thin-walled hollow sphere

D. two or more of these are tied for fastest
A solid uniform spherical boulder rolls down a hill from a height $H$. The top half of the hill is rough, with sufficient friction that the boulder rolls without slipping. The second half the hill is smooth with negligible friction. What is the translational speed of the boulder at the bottom of the hill?

\[ v = \sqrt{\frac{2}{5} g H} \]

**Example**

Remember:

\[
\begin{align*}
K_{\text{trans}} &= \frac{1}{2} \left( m v^2 \right) \\
\nu_{\text{tan}} &= \alpha R \\
K_{\text{rot}} &= \frac{1}{2} I \omega^2 \\
I_{\text{sphere}} &= \frac{2}{5} MR^2 \\
K_{\text{trans},i} + K_{\text{rot},i} + U_i + W_{\text{other}} &= K_{\text{trans},f} + K_{\text{rot},f} + U_f
\end{align*}
\]
You want to double the radius of a rotating solid sphere while keeping its kinetic energy constant. (The mass does not change.)

To do this, the final angular velocity of the sphere must be

A. 4 times its initial value.

B. twice its initial value.

C. the same as its initial value.

D. 1/2 of its initial value.

E. 1/4 of its initial value.

Remember:

\[ I_{sphere} = \frac{2}{5} MR^2 \]

\[ K_{rot} = \frac{1}{2} I \omega^2 \]