1 Problem 6.2

A system is defined by the wavefunction:

\[ \psi(x) = A \cos \left( \frac{2\pi x}{L} \right) \quad \text{for} \quad -\frac{L}{4} \leq x \leq \frac{L}{4} \]

(a) Determine the normalization constant A.
(b) What is the probability that the particle will be found between \( x = 0 \) and \( x = \frac{L}{8} \)?

1.1 Solution

1.1.1 Part (a)

\[ 1 = \int_{-\infty}^{+\infty} \psi^*(x) \psi(x) dx \]

\[ 1 = \int_{-\frac{L}{8}}^{\frac{L}{8}} 0 dx + \int_{-\frac{L}{4}}^{\frac{L}{4}} A^2 \cos^2 \left( \frac{2\pi x}{L} \right) dx + \int_{\frac{L}{8}}^{\infty} 0 dx + \]

\[ 1 = A^2 \int_{-\frac{L}{4}}^{\frac{L}{4}} \cos^2 \left( \frac{2\pi x}{L} \right) dx \]

\[ 1 = A^2 \frac{L}{4} \]

\[ A^2 = \frac{4}{L} \]

1.2 Part (b)

\[ P(0, \frac{L}{8}) = \int_0^{\frac{L}{8}} \psi^*(x) \psi(x) dx \]

\[ P(0, \frac{L}{8}) = \int_0^{\frac{L}{8}} \frac{4}{L} \cos^2 \left( \frac{2\pi x}{L} \right) dx \]

\[ P(0, \frac{L}{8}) = \frac{4}{L} \left( \frac{(2 + \pi)L}{16\pi} \right) \]

\[ P(0, \frac{L}{8}) = \frac{2 + \pi}{4\pi} \approx 0.41 \]
2 Problem 6.5

A particle with zero energy has a wavefunction

\[ \psi(x) = Axe^{-\frac{x^2}{L^2}} \]

Find and sketch \( V(x) \).

2.1 Solution

Start with the Schrodinger Equation.

\[ -\frac{\hbar^2}{2m} \frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x) \]

Notice that the particle has zero energy.

\[ \frac{\hbar^2}{2m} \frac{d^2\psi(x)}{dx^2} = V(x)\psi(x) \]  \( \text{(1)} \)

Now to figure out the second derivative of the wavefunction. Make sure to use the chain rule!

\[ \frac{d\psi(x)}{dx} = A(1)e^{-\frac{x^2}{L^2}} + Ax \left( -\frac{2x}{L^2} \right) e^{-\frac{x^2}{L^2}} \]

\[ \frac{d\psi(x)}{dx} = A \left( 1 - \frac{2x^2}{L^2} \right) e^{-\frac{x^2}{L^2}} \]

\[ \frac{d^2\psi(x)}{dx^2} = A \left( -\frac{4x}{L^2} \right) e^{-\frac{x^2}{L^2}} + A \left( 1 - \frac{2x^2}{L^2} \right) \left( -\frac{2x}{L^2} \right) e^{-\frac{x^2}{L^2}} \]

\[ \frac{d^2\psi(x)}{dx^2} = A \left( \frac{4x^3}{L^4} - \frac{6x}{L^2} \right) e^{-\frac{x^2}{L^2}} \]

Now we can plug this and the original wavefunction back into Equation 1.

\[ \frac{\hbar^2}{2m} A \left( \frac{4x^3}{L^4} - \frac{6x}{L^2} \right) e^{-\frac{x^2}{L^2}} = V(x)Axe^{-\frac{x^2}{L^2}} \]

Now we can cancel a bunch of terms out to get

\[ V(x) = \frac{\hbar^2}{2m} \left( \frac{4x^2}{L^4} - \frac{6}{L^2} \right) \]

This is an oscillator’s potential.
3 Problem 6.6

The wavefunction of a particle is given as

$$\psi(x) = A \cos(kx) + B \sin(kx)$$

Show that it is a solution is the Schrodinger Equation with $V(x)= 0$ and find the energy.

3.1 Solution

Start with the Schrodinger Equation.

$$-\frac{\hbar^2}{2m} \frac{d^2 \psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x)$$

There’s no potential so

$$-\frac{\hbar^2}{2m} \frac{d^2 \psi(x)}{dx^2} = E\psi(x)$$ (2)

Calculate the second derivative of the wavefunction

$$\frac{d\psi(x)}{dx} = -A k \sin(kx) + B k \cos(kx)$$

$$\frac{d^2 \psi(x)}{dx^2} = -k^2 (A \cos(kx) + B \sin(kx))$$

Plug this back into Equation 2

$$\frac{\hbar^2 k^2}{2m} [A \cos(kx) + B \sin(kx)] = E [A \cos(kx) + B \sin(kx)]$$

So this is a solution to the Schrodinger Equation provided we have

$$E = \frac{\hbar^2 k^2}{2m} = \frac{p^2}{2m}$$

Which is exactly what we would expect for a free particle.

4 Problem 6.11

A particle is confined to a box from $-\frac{L}{2} \leq x \leq \frac{L}{2}$

What are the wavefunctions and probability densities for the $n = 1, 2, \text{ and } 3$ states? Sketch them.
4.1 Solution

The general solution for a particle in a box is still:

$$\psi(x) = A \sin(kx) + B \cos(kx)$$

But now the boundary conditions are

$$\psi\left(-\frac{L}{2}\right) = 0; \psi\left(\frac{L}{2}\right) = 0$$

The first of these tells us

$$0 = -A \sin(kL/2) + B \cos(kL/2)$$

The second of these tells us

$$0 = A \sin(kL/2) + B \cos(kL/2)$$

The difference in sign is due to \(\cos(x)\) being symmetric and \(\sin(x)\) being antisymmetric. Equating the two gives

$$A \sin(kL/2) = -A \sin(kL/2)$$

But this only makes sense if \(A = 0\), so the wavefunction is

$$\psi(x) = B \cos(kx)$$

Returning to the boundary conditions we have

$$0 = B \cos(kL/2)$$

This means that

$$\frac{kL}{2} = (n + \frac{1}{2})\pi$$

So

$$k = \frac{\pi}{L}(2n + 1)$$

This makes the wavefunction

$$\psi_n(x) = A \cos\left(\frac{x\pi}{L}(2n + 1)\right)$$

Normalizing will still yield

$$A = \sqrt{2/L}$$

So

$$\psi_n(x) = \sqrt{2/L} \cos\left(\frac{x\pi}{L}(2n + 1)\right)$$
This means the probability density will be

\[ P_n(x) = \frac{2}{L} \cos^2 \left( \frac{x\pi}{L} (2n + 1) \right) \]

5 Problem 6.24

A solution of Schrodinger’s equation for an oscillator is

\[ \psi(x) = Cxe^{-\alpha x^2} \]

(a) Express \( \alpha \) in terms of \( m \) and \( \omega \). What is the energy of this state?
(b) Normalize it.

5.1 Solution

5.1.1 Part (a)

As always, start with the Schrodinger equation:

\[ \frac{-\hbar^2}{2m} \frac{d^2\psi(x)}{dx^2} + \frac{1}{2} kx^2 \psi(x) = E\psi(x) \]

Notice that this wavefunction is essentially the same as the one from problem 6.5 except we replaced \( \frac{1}{L^2} \) with \( \alpha \). This means it’s second derivative will be:

\[ \frac{d^2\psi(x)}{dx^2} = C \left( 4\alpha^2 x^3 - 6\alpha x \right) e^{-\alpha x^2} \]

If we plug this into the Schrodinger Equation we get

\[ \frac{-\hbar^2}{2m} C \left( 4\alpha^2 x^3 - 6\alpha x \right) e^{-\alpha x^2} + \frac{1}{2} kx^2 Cxe^{-\alpha x^2} = E Cxe^{-\alpha x^2} \]

This simplifies to

\[ \frac{-\hbar^2}{2m} \left( 4\alpha^2 x^2 - 6\alpha \right) + \frac{1}{2} kx^2 = E \]  \( \text{(3)} \)

We can see that the \( x^2 \) terms must cancel each other out, giving

\[ \frac{\hbar^2}{2m} 4\alpha^2 x^2 = \frac{1}{2} kx^2 \]

Solving for alpha gives

\[ \alpha = \sqrt{\frac{km}{4\hbar^2}} \]

Remember that

\[ \omega = \sqrt{\frac{k}{m}} \]
Plugging this in gives

\[ \alpha = \frac{m\omega}{2\hbar} \]

The other consequence of Equation 3 is

\[ -\frac{\hbar^2}{2m} (-6\alpha) = E \]

This simplifies to

\[ E = \frac{3\hbar^2\alpha}{m} \]

Plugging in alpha gives

\[ E = \frac{3}{2} \hbar \omega \]

Notice that this is the energy of a quantum harmonic oscillator in the n=1 state!

5.1.2 Part (b)

We have the standard rule for normalization

\[ 1 = \int_{-\infty}^{+\infty} \psi^*(x)\psi(x)dx \]

\[ 1 = \int_{-\infty}^{+\infty} C^2 x^2 e^{-2\alpha x^2} dx \]

\[ 1 = C^2 \left( \frac{1}{4} \sqrt{\frac{\pi}{2}} \alpha^{-3/2} \right) \]

\[ C^2 = 4 \sqrt{\frac{2}{\pi}} \alpha^{3/2} = 4 \sqrt{\frac{2}{\pi}} \left( \frac{m\omega}{2\hbar} \right)^{3/2} = \frac{2}{\sqrt{\pi}} \left( \frac{m\omega}{\hbar} \right)^{3/2} \]

6 Problem 6.29

An electron is bound to \( x > 0 \) with the wavefunction

\[ \psi(x) = Ce^{-x} (1 - e^{-x}) \]

(a) Normalize the wavefunction.
(b) What is the most probable value of \( x \)?
(c) What is the expectation value of \( x \)?
6.1 Solution

6.1.1 Part (a)

Again start with

\[ 1 = \int_{-\infty}^{+\infty} \psi^*(x)\psi(x)dx \]

\[ 1 = \int_{0}^{+\infty} C^2 e^{-2x} (1 - e^{-x})^2 \]

\[ 1 = C^2 \left( \frac{1}{12} \right) \]

\[ C^2 = 12 \]

6.2 Part (b)

We find the most probable value by finding where the derivative of the probability is 0:

\[ 0 = \frac{d}{dx} 12 \left( e^{-2x} (1 - e^{-x})^2 \right) \]

We can throw away the normalization constant, but must make sure to use the chain rule

\[ 0 = (-2)e^{-2x}(1 - e^{-x})^2 + e^{-2x}(2)(1 - e^{-x})^1(e^{-x}) \]

Canceling terms and factoring gives

\[ 0 = (2e^{-x} - 1)(1 - e^{-x}) \]

The term in the right parentheses is only 0 when \( x = 0 \), but that is outside the allowed range. This means

\[ 0 = 2e^{-x} - 1 \]

Which has the solution

\[ x = \ln 2 \approx 0.69 \]
6.3 Part (c)

To find the expectation value, we use

\[ \langle \hat{A} \rangle = \int_{-\infty}^{+\infty} \psi^*(x) \hat{A} \psi(x) \, dx \]

In our case this is:

\[ \langle x \rangle = 12 \int_{0}^{+\infty} x e^{-2x} (1 - e^{-x})^2 \, dx \]

This works out to be

\[ \langle x \rangle = \frac{13}{12} \approx 1.1 \]

This means that the most probable value is NOT the expectation value. This is because the expectation value is just the average value. The most probable value is just the highest point on the plot of \( P(x) \) vs \( x \). If you have an asymmetric curve, these two are not necessarily the same.

7 Problem 6.35

Which of the following are eigenfunctions of the momentum operator \( \hat{p} = i\hbar \frac{d}{dx} \)

(a) \( \psi(x) = A \sin(kx) \)
(b) \( \psi(x) = A \sin(kx) - A \cos(kx) \)
(c) \( \psi(x) = A \cos(kx) + iA \sin(kx) \)
(d) \( \psi(x) = Ae^{ik(x-a)} \)

7.1 Solution

The generic eigenvalue problem for an operator \( \hat{A} \) is

\[ \hat{A} \psi = c \psi \]

Where \( c \) is a constant and an eigenvalue of \( \hat{A} \). In our case \( \hat{A} = \hat{p} = i\hbar \frac{d}{dx} \)

7.2 Part (a)

\[ i\hbar \frac{d}{dx}(A \sin(kx)) = i\hbar Ak \cos(kx) \neq c(A \sin(kx)) \]

So it is NOT an eigenfunction.
7.3 Part (b)
\[ i\hbar \frac{d}{dx} (A \sin(kx) - A \cos(kx)) = i\hbar k (A \sin(kx) + A \cos(kx)) \neq c (A \sin(kx) - A \cos(kx)) \]
So it is NOT an eigenfunction.

7.4 Part (c)
\[ i\hbar \frac{d}{dx} (A \cos(kx) + iA \sin(kx)) = i\hbar (-Ak \sin(kx) + ikA \cos(kx)) = -\hbar k (A \cos(kx) + iA \sin(kx)) \]
So this IS an eigenvector with eigenvalue
\[ c = -\hbar k = -p \]

7.5 Part (d)
\[ i\hbar \frac{d}{dx} (e^{ik(x-a)}) = i\hbar (ik) e^{ik(x-a)} = -\hbar k (e^{ik(x-a)}) \]
So this IS an eigenvector with eigenvalue
\[ c = -\hbar k = -p \]

8 Additional Problem
8.1 Solution
The wavefunction is:
\[ \psi(x) = \left[ \frac{2}{\sqrt{\pi}} \left( \frac{m\omega}{\hbar} \right)^{3/2} \right]^{1/2} x^3 e^{-\alpha x^2} \]
With
\[ \alpha = \frac{m\omega}{2\hbar} \]

8.1.1 Part (a)
\[ \langle x \rangle = \int_{-\infty}^{\infty} \psi^* (x) \hat{x} \psi (x) dx \]
\[ \langle x \rangle = \frac{2}{\sqrt{\pi}} \left( \frac{m\omega}{\hbar} \right)^{3/2} \int_{-\infty}^{\infty} x^3 e^{-2\alpha x^2} dx \]
\[ \langle x \rangle = 0 \]
Since \( x^3 \) is odd (antisymmetric) and \( e^{-2\alpha x^2} \) is even (symmetric), their product is odd. Integrating an odd function over an even interval will give zero.
8.1.2 Part (b)

\[
\langle x^2 \rangle = \int_{-\infty}^{+\infty} \psi^*(x)x^2\psi(x)dx
\]

\[
\langle x^2 \rangle = \frac{2}{\sqrt{\pi}} \left( \frac{m\omega}{\hbar} \right)^{3/2} \int_{-\infty}^{+\infty} x^4e^{-2\alpha x^2}dx
\]

\[
\langle x^2 \rangle = \frac{2}{\sqrt{\pi}} \left( \frac{m\omega}{\hbar} \right)^{3/2} \left( \frac{3}{16} \sqrt{\frac{\pi}{2}} \alpha^{-5/2} \right)
\]

\[
\langle x^2 \rangle = \frac{3}{2} \frac{\hbar}{m\omega}
\]

8.1.3 Part (c)

\[
\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\langle x^2 \rangle}
\]

\[
\Delta x = \sqrt{\frac{3}{2} \frac{\hbar}{m\omega}}
\]

8.1.4 Part(d)

\[
\langle p \rangle = \int_{-\infty}^{+\infty} \psi^*(x)p\psi(x)dx
\]

\[
\langle p \rangle = \frac{2}{\sqrt{\pi}} \left( \frac{m\omega}{\hbar} \right)^{3/2} \int_{-\infty}^{+\infty} x e^{-\alpha x^2} \left(-i\hbar \frac{d}{dx}\right) x e^{-\alpha x^2} dx
\]

\[
\langle p \rangle = \frac{2}{\sqrt{\pi}} \left( \frac{m\omega}{\hbar} \right)^{3/2} \int_{-\infty}^{+\infty} x e^{-\alpha x^2} (-i\hbar) (1 - 2\alpha x^2) e^{-\alpha x^2} dx
\]

These will both lead to odd functions being integrated over an even interval, so we know it will go to zero.

\[
\langle p \rangle = 0
\]

8.1.5 Part (e)

\[
\langle p^2 \rangle = \int_{-\infty}^{+\infty} \psi^*(x)p^2\psi(x)dx
\]

\[
\langle p^2 \rangle = \frac{2\hbar^2}{\sqrt{\pi}} \left( \frac{m\omega}{\hbar} \right)^{3/2} \int_{-\infty}^{+\infty} x^2 e^{-\alpha x^2} \left(-i\hbar \frac{d}{dx}\right)^2 x e^{-\alpha x^2} dx
\]

We already calculated the second derivative of this wavefunction, so we can just plug it in

\[
\langle p^2 \rangle = -\frac{2\hbar^2}{\sqrt{\pi}} \left( \frac{m\omega}{\hbar} \right)^{3/2} \int_{-\infty}^{+\infty} x^2 e^{-\alpha x^2} \left(4\alpha^2 x^3 - 6\alpha x\right) e^{-\alpha x^2} dx
\]
\begin{align*}
\langle p^2 \rangle &= -\frac{2\hbar^2}{\sqrt{\pi}} \left( \frac{m\omega}{\hbar} \right)^{3/2} \left[ 4\alpha^2 \int_{-\infty}^{\infty} x^4 e^{-2\alpha x^2} \, dx + (-6\alpha) \int_{-\infty}^{\infty} x^2 e^{-2\alpha x^2} \, dx \right] \\
\langle p^2 \rangle &= -\frac{2\hbar^2}{\sqrt{\pi}} \left( \frac{m\omega}{\hbar} \right)^{3/2} \left[ 4\alpha^2 \left( \frac{3}{16} \sqrt{\frac{\pi}{2}} \alpha^{-5/2} \right) - 6\alpha \left( \frac{1}{4} \sqrt{\frac{\pi}{2}} \alpha^{-3/2} \right) \right] \\
\langle p^2 \rangle &= -\frac{2\hbar^2}{\sqrt{\pi}} \left( \frac{m\omega}{\hbar} \right)^{3/2} \alpha^{-1/2} \sqrt{\frac{\pi}{2}} \left[ \left( \frac{3}{4} \right) - \left( \frac{6}{4} \right) \right] \\
\langle p^2 \rangle &= \frac{3}{2} m\omega \hbar
\end{align*}

8.1.6 Part (f)

\[ \Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \sqrt{\langle p^2 \rangle} \]

\[ \Delta p = \sqrt{\frac{3}{2} m\omega \hbar} \]

8.1.7 Part (g)

\[ \Delta x \Delta p = \sqrt{\frac{3}{2} \hbar} \sqrt{\frac{3}{2} m\omega \hbar} \]

\[ \Delta x \Delta p = \frac{3}{2} \hbar \]

This obeys the Heisenberg uncertainty principle, which says that \( \Delta x \Delta p \geq \frac{\hbar}{2} \).

If you repeat this for an \( n = 0 \) quantum harmonic oscillator, you will actually get the minimum value of \( \Delta x \Delta p = \frac{\hbar}{2} \).