Work and Energy

• This is Fundamental to Engineering

Definition of Work

\[ W = \vec{F} \cdot \vec{d} \]
Work done and Work experienced

• **Something subtle:** The amount of work you do on a body may not be the same as the work done on a body.

• Only the **NET** force on the object is used in the total work calculation.

• Add up all the work done on an object to find the **total** work done!
Examples

• Holding a bag of groceries
  – Is it heavy?
  – Will you get tired holding it?
  – Are you doing “Work?”

• Moving a bag of groceries with constant speed across a room
  – Is it heavy?
  – Will you get tired doing it?
  – Are you doing “Work?”

• Lifting a bag of groceries a height $h$ with constant speed?
  – Work by you?
  – Work on the bag?
Hiker with a backpack

Determine:

• The work done by the hiker
• The work done by gravity
• The work on the backpack
\[ W_p = F_p \cdot d = F_p \cdot d \cdot \cos \Theta = F_p \cdot h \]

\[ W_g = m\vec{g} \cdot \vec{d} = mgd \cdot \cos (180^\circ - \Theta) = mgd (1 + \cos \Theta) = mg (h - d \sin \Theta) \]

\[ w_g = -mgh \]

\[ F_p = mg \Rightarrow w_p = mgh \]

\[ W_{total} = 0 \]
Does the Earth do work on the Moon?
Work by Integration

To find the total work, we must sum up all the little pieces of work (i.e., $Fd$). If the force is continually changing, then we have to take smaller and smaller lengths to add. In the limit, this sum becomes an integral.

$$\int_{a}^{b} \mathbf{F}_i \cdot d\mathbf{x}$$
Work and Energy

\[ W = \vec{F} \cdot \vec{d} \]

\[ W = \Delta \Delta K \]
Force to stop a car

What constant net force is required to bring a car of mass $m$ to rest from a speed of $V$ within a distance of $D$?

\[ V_0 = V \]

\[ X_0 = 0 \]

\[ V = 0 \]

\[ X_F = D \]
Force to stop a car

\[ v^2 = v_0^2 + 2a(x - x_0) \]

\[ 0 = v_0^2 + 2ad \]

\[ a = -\frac{v_0^2}{2d} \]

\[ F = ma = -m \frac{v_0^2}{2d} \]

\[ W = Fd = -m \frac{v_0^2}{2} \]

\[ K = \frac{mv^2}{2} \]

\[ W = \Delta K \]
Kicking a ball

\[ F = \frac{m \cdot v^2}{2d} = 900 \text{ N} \]
Challenge Problem. An earth-penetrating warhead (EPW) has a mass $m=400$ kg and is designed to penetrate into the ground before exploding. Assume that this warhead hits the ground at a velocity of 100 m/s (vertically down), and the force of ground resistance is given by $F=F_0(1+y/D)$, where $y$ is the depth into the ground, $F_0=10^5$ N, and $D=1$ m. How deep will the warhead penetrate before it is stopped by the resistance of ground?

\[ W = \int F \, dy = \int F_0 \left(1 + \frac{y}{D}\right) \, dy = \]

\[ = F_0 \left(y + \frac{y^2}{2D}\right) \bigg|_0^Y = \]

\[ = F_0 \left(Y + \frac{Y^2}{2D}\right) - 0 \]

\[ W = \Delta K = \frac{m \, v_0^2}{2} \]

\[ F_0 \left(Y + \frac{Y^2}{2D}\right) = \frac{m \, v_0^2}{2} = \frac{400 \text{ kg} \cdot (100 \text{ m})^2}{2} = \frac{4 \cdot 10^7}{2} = 2 \cdot 10^6 \text{ J} \]

\[ 10^5 N \left(Y + \frac{Y^2}{2 \cdot 1 \text{ m}}\right) = 2 \cdot 10^6 \text{ J} \]

\[ Y = 19 \text{ m} \]
Water Slide

Who hits the bottom with a faster speed?

Paul

Kathleen

$h$
Multiple ways to calculate work

1. If the force and direction is constant
   - $\vec{F} \cdot \vec{d}$

2. If the force isn’t constant
   - Integrate

3. If we don’t know much about the forces
   - Look at the change in kinetic energy
Potential Energy

\[ U = W \]

Is a measure of work that a system *COULD* do.
Last time we considered a hiker carrying a backpack of mass $M$ with constant speed up a hill of angle $\Theta$ and height $h$.

We have determined:

• The work done by the hiker

• The work done by gravity

• The work on the backpack
Gravitational Potential Energy

I lift up a brick from the ground to a height $h$, and let it go. How much work will be done by gravity by the time the brick hits the ground?

$$U = mgh$$

Subtlety: Potential energy is relative to somewhere!
Example: What is the potential energy of a brick 2 meters above a 1 meter high table? 3 meters above the floor?

Only change in potential energy is really meaningful!
Springs

- Hooke’s Law

\[ \vec{F} = -k\vec{x} \]

Some constant
Displacement

- Force is NOT CONSTANT over a distance
Work by a spring

\[ \vec{F}_s = -k \vec{x} \]

\[ \vec{F}_p = -\vec{F}_s = k \vec{x} \]

\[ W_p = \int \vec{F}(x) \cdot d\vec{x} \]

\[ W_p = \int k \vec{x} \ d\vec{x} = \int kx \ dx \]

\[ = \left. \frac{kx^2}{2} \right|_0^D \]

\[ W_p = \frac{kD^2}{2} \]
Potential Energy of a Spring

How much work do we exert to compress a spring by a distance $x$?

How much potential energy does it now have?

$U(x) = \frac{1}{2}kx^2$
Mechanical Energy

\[ E = K + U \]

For some types of problems, Mechanical Energy is conserved:

\[ K_2 + U_2 = K_1 + U_1 \]
Problem Solving

As objects move, their potential energy gets converted into kinetic energy, and back.

It’s often useful to draw “before” and “after” diagrams.
"before"

\[ h \]

\[ u = mgh \]

\[ k = 0 \]

"after"

\[ O \downarrow v \]

\[ u' = 0 \]

\[ k = \frac{mv'^2}{2} \]

\[ u + k = u' + k' \]

\[ mgh = \frac{mv'^2}{2} \]

\[ v = \sqrt{2gh} \]
**Before**

\[ U_0 = \text{mgh} \]
\[ K_0 = \frac{\text{m} \cdot U_0^2}{2} \]

**after**

\[ U_f = \text{mg} \cdot \theta \]
\[ K_f = \frac{\text{m} \cdot U_f^2}{2} \]

\[ U_0 + K_0 = U_f + K_f \]

\[ 2 \cdot \text{mgh} + \frac{\text{m} \cdot U_0^2}{2} = \frac{\text{m} \cdot U_f^2}{2} \]

\[ U_f = \sqrt{U_0^2 + 2gh} \]
Rollercoaster Problem

A Roller Coaster of mass $M$ starts at the top, height $h$, with an initial speed $V_0=0$.

a) What is the energy at the top?

b) What is the speed at the bottom?

c) How much work is done by gravity in going from the top to the bottom?

d) At what height is it at half the max speed?
Rollercoaster

(a) \( U_1 = mgh \)

(b) \( \frac{m \cdot v_b^2}{2} = mgh \Rightarrow v_b = \sqrt{2gh} \)

(c) \( W = U_1 = mgh \)

(d) \( \frac{m \cdot v^2}{2} = mg(h-y) \Rightarrow \frac{m \cdot gh}{2} = mg(h-y) \)

\( v = \frac{1}{2} v_b \)

\( v^2 = \frac{v_b^2}{4} = \left( \frac{\sqrt{2gh}}{4} \right)^2 = \frac{gh}{2} \)

\( \frac{h}{4} = h-y \)

\( y = \frac{3}{4} h \)
If there were no friction, the roller coaster would go back up to height $h$. Would it go as high, if there were friction?

Three different types of forces acting:

– Gravity: *Conserves mechanical energy*
– Normal Force: *Conserves mechanical energy*
– Friction: *Doesn’t conserve mechanical energy*

Friction is a *Non-Conservative force*!
Energy Conservation

Friction turns mechanical energy into heat

\[ E = \text{Kinetic Energy} + \text{Potential Energy} + \]
\[ + \text{Heat} + \text{Others} \ldots \quad -- \quad \text{This is what is really conserved!} \]

One can use “lost” mechanical energy to estimate things about friction
Emma Noether  
(23 March 1882 – 14 April 1935)

Noether's (first) theorem states that any differentiable symmetry of the action of a physical system has a corresponding conservation law.
$E_{\text{Heat+Light+Sound}} = -W_{\text{NC}}$

If work is done by a non-conservative force, take this into account in the total energy. (Friction causes mechanical energy to be lost.)

$K_1 + U_1 = K_2 + U_2 + E_{\text{Heat}}$

$K_1 + U_1 = K_2 + U_2 - W_{\text{NC}}$
Roller Coaster Again!

A roller coaster of mass $m$ starts at rest at height $y_1$ and falls down the path, then back up until it hits height $y_2$ ($y_1 > y_2$). An odometer tells us that the total scalar distance traveled is $d$.

Assuming we don’t know anything about the friction or the path, how much work is done by friction on this path?

Assuming that the magnitude of the force of friction, $F$, between the car and the track is constant, find $F$. 
"before"

\[ m, \ v_0 = 0 \]
\[ u_1 = mg y_1 \]

"after"

\[ m, \ v_f = 0 \]
\[ u_2 = mg y_2 \]

\[ mgy_1 = mgy_2 - w_{fr} \]

\[ w_{fr} = -mg(y_1 - y_2) \]

\[ w_{fr} = F_{fr} \cdot d = F_{fr} \cdot d \cdot \cos \theta = -F_{fr} \cdot d \]

\[ -F_{fr}d = -mg(y_1 - y_2) \]

\[ F_{fr} = \frac{mg}{d} (y_1 - y_2) \]
Force and Potential Energy

If we know the potential energy, $U(x)$, we can find the force

$$F_x = -\frac{dU}{dx}$$
Potential Energy Diagrams

• For Conservative forces we can draw energy diagrams

• Equilibrium points:
  – If placed there with no energy, an object will just stay (no force)
Stable vs. Unstable Equilibrium Points

The force is zero at both maxima and minima.
A force is conservative if the work done by a force on an object moving from one point to another point depends only on the initial and final positions and is independent of the particular path taken.

Another definition:
A force is conservative if the net work done by the force on an object moving around any closed path is zero.

This definition is equivalent to the previous one.
Friction a Non-Conservative Force

Is the force, applied by the person who moves the box, conservative?
A robot arm has a funny Force equation in 1-dimension

\[ F(x) = F_0 \left(1 + \frac{3x^2}{X_0^2}\right) \]

where \( F_0 \) and \( X_0 \) are constants.

What is the work done to move a block from position \( X_1 \) to position \( X_2 \)?
Problem

We drop a ball of mass $m$ from a height $h$ above the uncompressed spring and observe that it compresses a distance $Y$. What is $k$?
"before"

\[ h \]
\[ y \]
\[ 0 \]
\[ v=0 \]

\[ U_{g1} = mgh \]
\[ U_{s1} = 0 \]
\[ K_1 = 0 \]

"after"

\[ y \]
\[ Y \}
\[ 0 \}
\[ v=0 \]

\[ U_{g2} = -mgY \]
\[ U_{s2} = \frac{1}{2} kY^2 \]
\[ K_2 = 0 \]

\[ U_{g1} + U_{s1} + K_1 = U_{g2} + U_{s2} + K_2 \]

\[ mgh = -mgY + \frac{1}{2} kY^2 \]

\[ k = \frac{2}{Y^2} (h+Y) mg \]