R-9.5 Consider $\vec{a} = (-2, 1, 0)$, which is a vector. Then $\psi^* = (2, 1, 0)$. Under a 180° rotation about the y-axis, $\vec{a} \rightarrow \vec{a}' = (2, 1, 0)$ and therefore $\psi^* \rightarrow \psi^{*'} = (2, 1, 0) = \psi^*$. Therefore $\psi^*$ is unchanged. Now the angle between $\vec{a}$ and $\psi^*$ is clearly nonzero, but the angle between $\vec{a}'$ and $\psi^{*'}$ is zero. Since this angle is not preserved, either $\vec{a}$ or $\psi^*$ is not a vector. Since $\vec{a}$ is a vector, $\psi^*$ is not a vector under rotations.

R-10.1 (a) Since its direction is given by the right-hand rule (as with the vector product) and the vector product transforms as a vector under rotations, the direction of $\vec{a} \otimes \vec{b}$ transforms under rotation like a vector. Also since $|\vec{a}|, |\vec{b}|$ and $\theta_{\vec{a}, \vec{b}}$ are preserved under a rotation, so is $|\vec{a} \otimes \vec{b}| = |\vec{a}| |\vec{b}| \sin 2\theta_{\vec{a}, \vec{b}}$. Therefore $\vec{a} \otimes \vec{b}$ transforms as a vector under rotation.

(b) Let $\vec{a} = \hat{i}$, $\vec{b} = \hat{i}$, and $\vec{c} = \hat{j}$. Then

\[ |\vec{a} \otimes \vec{b}| = |\vec{i} \otimes \vec{i}| = |\vec{i}| |\vec{i}| |\sin(2 \cdot 0)| = 1 \cdot 1 \cdot 0 = 0, \]

\[ |\vec{a} \otimes \vec{c}| = |\vec{i} \otimes \vec{j}| = |\vec{i}| |\vec{j}| |\sin(2 \cdot \frac{\pi}{2})| = 1 \cdot 1 \cdot 0 = 0, \]

\[ |\vec{a} \otimes (\vec{b} + \vec{c})| = |\vec{i} \otimes (\vec{i} + \vec{j})| = |\vec{i}| |\vec{i} + \vec{j}| \sin \left(2 \cdot \frac{\pi}{4}\right) = 1 \cdot \sqrt{2} \cdot 1 = \sqrt{2}. \]

Therefore, by the right-hand rule,

$\vec{a} \otimes \vec{b} = 0$, $\vec{a} \otimes \vec{c} = 0$, and $\vec{a} \otimes (\vec{b} + \vec{c}) = \sqrt{2}\hat{k}$.

Hence the $\otimes$ operator is not distributive, i.e.

$\vec{a} \times (\vec{b} + \vec{c}) \neq \vec{a} \otimes \vec{b} + \vec{a} \otimes \vec{c}$, in general.

R-10.3 (a) $\vec{a} \rightarrow \lambda\vec{a}$, and $\vec{b} \rightarrow \lambda\vec{b}$ imply that

$\vec{a} \otimes \vec{b} \rightarrow (\lambda\vec{a}) \otimes (\lambda\vec{b}) = |\lambda\vec{a}| |\lambda\vec{b}| \cos (2\theta_{\vec{a}, \vec{b}}) = |\lambda|^2 |\vec{a}| |\vec{b}| \cos (2\theta_{\vec{a}, \vec{b}}).$

Also, $\vec{b} \otimes \vec{a} = |\vec{b}| |\vec{a}| \cdot \cos (2\theta_{\vec{a}, \vec{b}}) = \vec{a} \otimes \vec{b}$ because $\theta_{\vec{a}, \vec{b}} \equiv \theta_{\vec{b}, \vec{a}}$.

Thus $\otimes$ is bilinear and commutative. Rotations don’t change $\theta_{\vec{a}, \vec{b}}$ and $|\vec{a}|, |\vec{b}|$, so $\vec{a} \otimes \vec{b}$ is a scalar under rotation.
(b)  
\[ \vec{a} \odot (\vec{b} + \vec{c}) = i \odot (i + j - i + j) \]
\[ = i \odot 2j = 1 \cdot 2 \cdot \cos \theta = -2. \]

But  
\[ \vec{a} \odot \vec{b} + \vec{a} \odot \vec{c} = i \odot (i + j) + i \odot (-i + j) \]
\[ = 1 \cdot \sqrt{2} \cdot \cos (2 \cdot 45^\circ) + 1 \cdot \sqrt{2} \cdot \cos (2 \cdot 135^\circ) \]
\[ = 0 + 0 = 0. \]

Thus  
\[ \vec{a} \odot (\vec{b} + \vec{c}) \neq \vec{a} \odot \vec{b} + \vec{a} \odot \vec{c}. \]

R-10.5  
(a) Let  \( \theta \).  
On the other hand, if  \( \hat{a} = (\hat{a}_1, \hat{a}_2) \) and  \( \hat{a}_2 = (\cos \theta, \sin \theta) \) then  \( \hat{a} \times \hat{a}_2 \) points along  \( k \) and  \( \hat{a}_1 \times \hat{a}_2 = k (\cos \theta \cdot \sin \theta - \sin \theta \cdot \cos \theta) \). Thus  \[ |\hat{a}_1 \times \hat{a}_2| = \cos \theta \cdot \sin \theta - \sin \theta \cdot \cos \theta = \sin (\theta_1 \cdot \cos \theta_2) = \sin (\theta_2 - \theta). \]

(b) Letting  \( \theta_1 \rightarrow -\theta_1 \) in the previous result, and using  \[ \cos (-\theta_1) = \cos \theta_1 \text{ and } \sin (-\theta_1) = \sin \theta_1, \]
we find that  \[ \sin (\theta_1 + \theta_2) = \cos \theta_1 \cdot \sin \theta_2 + \sin \theta_1 \cdot \cos \theta_2. \]

(c) If  \( \theta_1 \rightarrow \frac{\pi}{2} - \theta_1 \), then  \[ \sin (\frac{\pi}{2} + \theta_1 - \frac{\pi}{2}) = -\sin (\frac{\pi}{2} - \theta_1 - \theta_2) = -\cos (\theta_1 + \theta_2). \]
\[ -\cos (\theta_1 + \theta_2) = \cos (\frac{\pi}{2} - \theta_1) \sin \theta_2 - \sin (\frac{\pi}{2} - \theta_1) \cos \theta_2 \]
\[ = \sin \theta_1 \sin \theta_2 - \cos \theta_1 \cos \theta_2, \]

or  \[ \cos (\theta_1 + \theta_2) = \cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2. \]

R-10.7  
See Figure R.14. Here are some preliminaries.

The unlabeled angles  \( \angle OPB, \angle BPC, \angle PCB, \) and  \( \angle BAO \) are  \( \frac{\pi}{2} - \beta, \)
\( \frac{\pi}{2} - \alpha, \frac{\pi}{2}, \) and  \( \frac{\pi}{2}, \) respectively. Since  \( OP \) has unit length, this implies  \( OB \) and  \( PB \) are  \( \sin (\frac{\pi}{2} - \beta) = \cos \beta \text{ and } \sin \beta, \)
respectively. Thus  \( OA = \cos \beta \sin (\frac{\pi}{2} - \alpha) = \cos \beta \cos \alpha \) and  \( PC = \sin \beta \sin \alpha, \) respectively.

Now project a vertical line from  \( P \) down to  \( OA, \) to intersect  \( OA \) at  \( T. \)
This forms a rectangular with sides  \( PCAT, \) so the horizontal sides  \( TA \) and  \( PC \) are equal. Hence  \( OA - TA = OA \quad PC = OT. \) Since  \( OT \)
is  \( \cos (\alpha + \beta), \) our previous results for  \( OA \) and  \( PC \) give  \( \cos (\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta. \)

R-10.8  
\( \vec{a} \equiv (3, -4, 2), \vec{b} \equiv (2, 6, -1). \)

(a)  
\[ |\vec{a}| = \sqrt{3^2 + (-4)^2 + 2^2} = \sqrt{29} = 5.385, \]
\[ |\vec{b}| = \sqrt{2^2 + 6^2 + (-1)^2} = \sqrt{41} = 6.403, \]
\[ \vec{a} \cdot \vec{b} = (3)(2) + (-4)(6) + (2)(-1) = -20. \]
Define  \( \vec{c} = \vec{a} \times \vec{b}, \) so
\[ c_x = a_y b_z - a_z b_y = (-4)(-1) - (2)(6) = -8, \]
\[ c_y = a_z b_x - a_x b_z = (2)(2) - (3)(-1) = 7, \]
\[ c_z = a_x b_y - a_y b_x = (3)(6) - (-4)(2) = 26. \]

Thus \( \vec{c} \equiv (-8, 7, 26) \), and \( |\vec{c}| = \sqrt{789} = 28.1 \).

(b) \( \hat{\vec{a}} = \frac{\vec{a}}{|\vec{a}|} \equiv (.557, -.743, .371), \hat{\vec{b}} = \frac{\vec{b}}{|\vec{b}|} \equiv (.312, .937, -.156), \)
\[ \hat{\vec{c}} = \frac{\vec{c}}{|\vec{c}|} \equiv (-.285, .249, .926). \]

(c) \( \cos \theta_{ab} = \frac{\hat{\vec{a}} \cdot \hat{\vec{b}}}{|\vec{a}| |\vec{b}|} = -.580. \) Then either
\[ \theta_{ab} = 125.45^\circ \equiv -234.55^\circ \) (we can add or subtract 360 degrees) or
\[ \theta_{ab} = -125.45^\circ \equiv 234.55^\circ. \] Moreover,
\[ \sin \theta_{ab} = \frac{|\hat{\vec{a}} \times \hat{\vec{b}}|}{|\vec{a}| |\vec{b}|} = \frac{|\vec{c}|}{(5.385)(6.403)} = .815. \]
Then \( \theta_{ab} = \pm 54.55^\circ \) or \( \pm 234.55^\circ. \) Only the latter pair is consistent with the calculation based on the cosine.

R-10.9 (a) We now rotate \( \vec{a} \equiv (3, -4, 2) \) to \( \vec{a}' \equiv (5, 0, 2) \). Because the z-component doesn’t change, this is a rotation about the z-axis. It is counterclockwise by 53.1°, because it is a 3-4-5 right triangle.
(We could also obtain the rotation angle by taking the dot product of \( \vec{a}_{\text{in-plane}} \) and \( \vec{a}'_{\text{in-plane}} \).) (Note: The angle between \( \vec{a} \) and \( \vec{a}' \) is irrelevant; it is associated with \( \vec{a} \times \vec{a}' \), not z.)

(b) To find the rotated \( \vec{b} \), which we’ll call \( \vec{b}' \), we note that the in-plane part of \( \vec{b} \) satisfies \( \vec{b}_{\text{in-plane}} \equiv (2, 6) \), putting it in the first quadrant. It has length \( |\vec{b}_{\text{in-plane}}| = \sqrt{40} = 6.325 \), and angle relative to the x-axis of \( \tan^{-1}(6/2) = \tan^{-1}(3) = 71.6^\circ. \) Thus
\[ (2, 6) \equiv \vec{b}_{\text{in-plane}} \equiv |\vec{b}_{\text{in-plane}}| (\cos 71.6^\circ, \sin 71.6^\circ). \]
To find \( \vec{b} \) under 53.1° counterclockwise rotation, replace 71.57° by 71.6 + 53.1 = 124.7°, so
\[ \vec{b}_{\text{in-plane}} \equiv |\vec{b}_{\text{in-plane}}| (\cos 124.7^\circ, \sin 124.7^\circ) = (-3.60, 5.20). \]
Thus \( \vec{b} \equiv (-3.60, 5.20, -1) \).

(c) To find the rotated \( \vec{a} \times \vec{b} \), which we’ll call \( (\vec{a} \times \vec{b})' \), we note that the in-plane part of \( \vec{a} \times \vec{b} \) satisfies \( (\vec{a} \times \vec{b})_{\text{in-plane}} \equiv (-8, 7) \), putting it
in the second quadrant. It has length \(|(\vec{a} \times \vec{b})_{\text{in-plane}}| = \sqrt{113} = 10.63\), and angle \(\arctan(-7/8) = -41.2 + 180 = 138.8^\circ\). (We add the 180 degrees to put it in the second quadrant. Calculators are stupid about this sort of thing.) Thus

\[ (-8, 7) \equiv (\vec{a} \times \vec{b})_{\text{in-plane}} \equiv |(\vec{a} \times \vec{b})_{\text{in-plane}}| (\cos 138.8, \sin 138.8). \]

To find \(\vec{a} \times \vec{b}\) under 53.1° counterclockwise rotation, replace 138.8° by 138.8° + 53.1° = 191.9°, so

\[ (\vec{a} \times \vec{b})'_{\text{in-plane}} \equiv |(\vec{a} \times \vec{b})_{\text{in-plane}}| (\cos 191.9, \sin 191.9) \]

\[ = (-10.40, -2.20). \]

Also, \((\vec{a} \times \vec{b})_z = a_x b_y - a_y b_x = 26\), by problem R-10.8. Thus \((\vec{a} \times \vec{b})' \equiv (-10.40, -2.20, 26)\).

(d) An explicit computation of \(\vec{a}' \times \vec{b}'\) gives

\[
\begin{align*}
    a_y'b_z' - a_z'b_y' &= (0)(-1) - (2)(5.2) = -10.40, \\
    a_x'b_z' - a_z'b_x' &= (2)(-3.6) - (5)(-1) = -2.20, \\
    a_x'b_y' - a_y'b_x' &= (5)(5.2) - (0)(-3.6) = 26.
\end{align*}
\]

(e) Comparison yields \((\vec{a} \times \vec{b})' = \vec{a}' \times \vec{b}' = (-10.40, -2.20, 26)\).

(f) It is easily verified that \(|\vec{a}| = |\vec{a}'| = 5.385\), and that

\[ |\vec{b}| = |\vec{b}'| = 6.403. \]

It is also easily verified that \(\vec{a} \cdot \vec{b} = \vec{a}' \cdot \vec{b}' = -20\) and that \(\vec{a} \times \vec{b} = |(\vec{a} \times \vec{b})'| = 28.1\).

(g) Both the scalar product and the vector product correspond to an angle of \(\pm 54.55^\circ\). Because the scalar product \(\vec{a} \cdot \vec{b}\) and the magnitudes of both versions of the vector product for \(\vec{a} \times \vec{b}\) do not change under rotation, the angle between \(\vec{a}'\) and \(\vec{b}'\) is the same as between \(\vec{a}\) and \(\vec{b}\).

(h) Yes, the vector lengths and angles transform as expected.

R-10.11

\[
\vec{\tau} = \vec{\mu} \times \vec{B} = (-\hat{i} + 2\hat{j}) \text{A-m}^2 \times (2\hat{i} - \hat{j} + 5\hat{k}) \text{N/A-m} \\
= \left[i(-10 + 0) - j(-5 - 0) + k(1 - 4)\right] \text{N-m} \\
\text{so } \vec{\tau} = \left[10\hat{i} + 5\hat{j} - 3\hat{k}\right] \text{N-m} = (10, 5, -3) \text{N-m}
\]
R-10.13

\[ d\vec{F} = 2.4 \, \text{A} \cdot (\vec{i} + 2\vec{j}) \, \text{A-m}^2 \times (2\vec{i} - \vec{j} + 5\vec{k}) \, \text{N/A-m} \]

so \(d\vec{F} = \vec{i} (4.8 \cdot 10^{-3} \, \text{N}) + \vec{j} (9.6 \cdot 10^{-3} \, \text{N})\)

or \(d\vec{F} = (4.8 \cdot 10^{-3} \, \text{N}, 9.6 \cdot 10^{-3} \, \text{N}, 0)\)

R-10.15 The determinant of the matrix described is

\[
\begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
a_x & a_y & a_z \\
b_x & b_y & b_z
\end{vmatrix} = (a_y b_z - a_z b_y) \hat{i} - (a_x b_z - a_z b_x) \hat{j} + (a_x b_y - a_y b_x) \hat{k}
\]

\[= \vec{a} \times \vec{b} \quad \text{by (R.31).}\]

R-10.17 Since the points \(\vec{r}_0\) and \(\vec{r}\) are in the plane, the line \(\vec{r} - \vec{r}_0\) is in the plane. Thus by definition it is normal to \(\hat{n}\), i.e. \((\vec{r} - \vec{r}_0) \cdot \hat{n} = 0\).

R-10.19 As indicated in the problem statement, \(\vec{r}_0 = (x_0, y_0, z_0)\) is not unique.

In particular, we can set \(x_0 = y_0 = 0\). Since \(\vec{r}_0\) is in the plane, it must satisfy

\[ax + by + cz + d = 0,\]

where \(a, b, c, d\) are constants.

Thus \(0 = cx_0 + by_0 + cz_0 + d = 0\), which implies \(z_0 = -d/c\). Notice that if we set \(\vec{n} = (a, b, c)\), then for any \(\vec{r} = (x, y, z)\), we have

\[(\vec{r} - \vec{r}_0) \cdot \vec{n} = (x, y, z + d/c) \cdot (a, b, c) = ax + by + cz + d.\]

Therefore, the equations \((\vec{r} - \vec{r}_0) \cdot \vec{n} = 0\) and \(ax + by + cz + d = 0\) are equivalent and define the same plane. Now set \(\hat{n} = \vec{n}/|\vec{n}|\). Then all points \(\vec{r}\) in the plane \(ax + by + cz + d = 0\) satisfy the equation \((\vec{r} - \vec{r}_0) \cdot \hat{n} = (\vec{r} - \vec{r}_0) \cdot \vec{n}/|\vec{n}| = 0\).

To summarize, if we define \(e = \sqrt{a^2 + b^2 + c^2}\), then we have shown that one can write \(\vec{r}_0 = (0, 0, -d/c)\) and \(\hat{n} = (a/e, b/e, c/e)\),

R-10.21 Our input is \(dA = 0.52 \, \text{mm}^2, \vec{r} = (-5, -2, 6) \, \text{m}\) (\(\vec{r}\) is irrelevant),
\(\vec{E} = (13, 27, -18) \, \text{V/m}\), and \(\hat{n}\) is along the vector \((2, -3, 7)\), of length \(\sqrt{2^2 + (-3)^2 + 7^2} = \sqrt{62} = 7.87\).

(a) \(\hat{n} \equiv \frac{(2, -3, 7)}{7.87} \equiv (0.254, -0.381, 0.889)\).

(b) \(\frac{d\Phi_E}{dA} = \vec{E} \cdot \hat{n} = -23.0 \, \text{V/m}\).

(c) \(d\Phi_E = \frac{d\Phi_E}{dA} dA = -1.195 \times 10^{-5} \, \text{V-m}\).