Irreversible Thermodynamics of Non-Uniform Insulating Ferromagnets

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Two types of damping are commonly applied to describe ferromagnets at the macroscopic level: one by Landau and Lifshitz, and the other by Gilbert. This work successively applies the methods of irreversible thermodynamics to insulators, insulating paramagnets, and uniform insulating ferromagnets, in the last case uniquely obtaining Landau-Lifshitz damping. These methods are then applied to non-uniform insulating ferromagnets, for which new spin fluxes and spin torques appear. These may be relevant to the dynamics of magnets in confined geometries, where boundary conditions impose complex non-trivial textures. For diffusion and damping we discuss the analogy between magnets and fluids confined within a porous medium.

I. INTRODUCTION

The present work is directed to two topics in the damping of insulating ferromagnets, both studied with the methods of irreversible thermodynamics: (1) damping in uniform ferromagnets, where two forms of damping are currently employed, one by Landau and Lifshitz (LL), and the other by Gilbert; (2) damping in non-uniform insulating ferromagnets, which becomes relevant for non-monodomain nanomagnets. Although damping of non-uniform conducting ferromagnets is timely, it is of value to clearly establish the effects of non-uniformity before considering the effects of conduction.

The theory of damping in a ferromagnet dates to the 1935 work of Landau and Lifshitz, who wrote down an equation of motion that included precession and damping, the latter intended to be phenomenological.\(^1\) Damping in paramagnetic systems was developed in the 1950's by a number of others, notably Bloch and co-workers. A number of phenomenological and microscopic theories were developed at that time, the one by Gilbert becoming the most popular, and currently in wide use.\(^2\) One aspect of the theory of Gilbert (but not of LL) is that it supports the possibility of critical damping.\(^3\) Nevertheless, the equations of motion of Landau and Lifshitz, and of Gilbert, are formally equivalent in the sense that, by redefining parameters for one theory one can get exactly the same equations as for the other theory.

Many experimentalists, and many theorists working on microscopic mechanisms, often choose to interpret their results for damping within the framework of Gilbert. Nevertheless there are strong reasons to prefer the framework of Landau and Lifshitz. First, it is a consequence of irreversible thermodynamics.\(^4,5\) Second, it has been derived both by projector operator techniques\(^6–8\) and by Langevin equation methods.\(^9\) Moreover, Landau-Lifshitz damping does not require that one renormalize the effective g-factor.

Landau and Lifshitz merely wrote down their form for the damping. At the time of their work, the theory of irreversible thermodynamics was still in its infancy. This was despite the early work of Einstein, Langevin, and others on near-equilibrium damping of particles in a fluid, and despite the work of Onsager on the symmetry of the kinetic coefficients that appear in phenomenological theories of damping for near-equilibrium macroscopic systems.\(^10\) Landau had not yet developed his theory of superfluidity of \(^4\)He.\(^11\) which later Khalatnikov systematized, implicitly using the methods of irreversible thermodynamics.\(^12\) The concept of the order parameter (\(\vec{M}\) for a ferromagnet) was simultaneously being developed by Landau, but the concept of broken symmetry (e.g., the direction of \(\vec{M}\) for a ferromagnet), a product of the early 1960's, was far in the future. The theory of irreversible thermodynamics, which applies to systems near equilibrium and subject to motions that vary slowly in space and time, relative to appropriate mean-free paths and collision times, was developed during the late 1940's and early 1950's.\(^13,14\)

In the late 1960's Halperin and Hohenberg applied the methods of irreversible thermodynamics to a number of broken symmetry systems, including isotropic ferromagnets and antiferromagnets.\(^15\) If the resulting equations lead to excitations for which the frequency \(\omega \to 0\) when the wavevector \(k \to 0\), as occurs for broken symmetry systems, then the resulting theory is said to be “hydrodynamic”. In the early 1970's a number of works used broken symmetry to develop the irreversible thermodynamics of liquid crystals.\(^16,17\) and superfluid \(^3\)He.\(^18–20\) Shortly thereafter a text applying irreversible thermodynamics to broken symmetry systems appeared;\(^21\) and the topic is treated in at least two recent texts.\(^22,23\) Irreversible thermodynamics has been applied to numerous other systems, among them supersolids,\(^24–26\) ferrofluids,\(^27\) spin glasses,\(^28,29\) and semiconductors (this last not being a broken symmetry system).\(^30\)

Despite some late 1960’s predictions of the demise of magnetism, research on this subject has, if anything, become more important, active, and widespread. Given that real ferromagnetism (with anisotropy and external magnetic fields, so there is no broken symmetry) is so relevant to real-world magnetic memory, the study and/or use of ferromagnets extends far beyond the borders of pure research. Nevertheless, the theory of damping in ferromagnets and paramagnets, using the principles of irreversible thermodynamics (which leads to Landau-Lifshitz damping) is not well-known, perhaps because of terseness.
of presentation. This may be a contributing factor to the continued use of the Gilbert form of damping.

As indicated above, the present work derives LL damping for an insulator using the methods of irreversible thermodynamics, both for its own sake and because it should precede studies of the electrical conduction aspect of non-uniform conducting ferromagnets. An ultimate motivation is study of the phenomenon of current-induced spin torque, whereby spin is transferred, by spin-polarized current, from one part of a magnetic texture to another. The formal equivalence of LL and Gilbert damping is retained even when, for a conductor, one includes current-induced spin torque. In particular, the present work emphasizes that LL damping is unavoidable within irreversible thermodynamics; Gilbert damping simply cannot occur within this framework. As noted above, this removes a difficulty associated with Gilbert damping, whereby the g-factor, a measure of the Zeeman interaction, is renormalized by damping effects. Moreover, as we show, the methods of irreversible thermodynamics readily yield expressions for damping in non-uniform ferromagnets, whereas there is no obvious means to do this using the approach of Gilbert.

An outline of this work is as follows. Sect. 2 discusses the basics of irreversible thermodynamics, and discusses the intrinsic time-reversal signature of extensive and intensive thermodynamic variables, and of their sources and fluxes, results which are applicable to all of the systems to be studied. Sect. 3 then considers insulators, insulating paramagnets, and uniform insulating ferromagnets, obtaining their equations of motion using the methods of irreversible thermodynamics. Sect. 4 discusses LL damping and Gilbert damping in more detail. Sect. 5 considers insulating ferromagnets with non-uniform magnetic textures (e.g. domain walls). To our knowledge, the methods of irreversible thermodynamics have not been applied previously to such systems. Even without a current (properly, without a gradient of the electrochemical potential) the theory predicts a number of new terms in the spin flux and in the spin torque, which should have an especially large effect on the dynamics of small (but not monodomain) magnets, where boundary conditions impose rapid spatial variation of textures. Sect. 6 presents a summary.

We note that this work serves as a precursor to the study of both conduction and magnetism.

II. BASICS OF IRREVERSIBLE THERMODYNAMICS

Basic terminology employed in irreversible thermodynamics are thermodynamic force, thermodynamic flux, thermodynamic source, intrinsic time-reversal signature, time-reversal signature, dissipative, and reactive. The essential idea behind irreversible thermodynamics is that the fluxes and sources are proportional to the forces, and whether the fluxes and sources are dissipative or reactive (non-dissipative), or a combination of the two, depends on the intrinsic time-reversal signature of the forces.

A. Intrinsic Time-Reversal Signature of Conjugate Thermodynamic Quantities

In thermodynamics, the energy density differential \( d\varepsilon \) can be written as a sum of pairs of terms. For each pair, one is an intensive thermodynamic quantity (e.g., temperature \( T \)) and the other is the differential of the density of an extensive thermodynamic quantity (e.g., entropy density \( s \)), with the intensive and extensive pair being thermodynamic conjugate. We now establish the simple but essential result that each term in the pair has a unique signature under time-reversal \( T \), and that for a given pair these signatures are the same. This is because the energy density must not change under time reversal.

For simplicity, let \( d\varepsilon = A dB + B dB \), where \( B \) is extensive and \( A \) is intensive. An example is \( T ds \), where \( T \) is the temperature and \( s \) is the entropy density. Now let

\[
A = A_E + A_O, \quad B = B_E + B_O, \quad (1)
\]

where the subscripts \( E \) and \( O \) denote even-ness and odd-ness under \( T \). Then

\[
d\varepsilon = [A_E dB_E + A_O dB_O] + [A_E dB_O + A_O dB_E]. \quad (2)
\]

The first two terms are both even under \( T \) and the last two terms are odd under \( T \). Because \( \varepsilon \) is even even under \( T \), the last two terms must disappear. The only way for this to happen is if \( A \) and \( B \) are both even or both odd.

For example, for a magnetic system, a term \(- \vec{H} \cdot d\vec{M}\) appears, with \( \vec{H} \) a magnetic field and \( \vec{M} \) a magnetization. Both are known to be odd under \( T \), in agreement with the general result given above.

In short, the time-reversal signatures of conjugate thermodynamic quantities are both well-defined and the same. That is, they are both either even or odd. We call these their intrinsic time-reversal properties. They could equally well be called their microscopic time-reversal properties.

B. Time-Reversal Signature of Fluxes and Sources Associated with Thermodynamic Quantities

Thermodynamic fluxes, by definition, appear in equations of motion in the form of a divergence. Likewise thermodynamic sources, by definition, appear in equations of motion not acted on by a divergence.

Consider the entropy density \( s \). It has both a flux \( j_i^s \) and a source \( R/T \), where \( R \) is the volume rate of heat production:

\[
\partial_t s + \partial_i j_i^s = \frac{R}{T} \geq 0. \quad (3)
\]
The intrinsic time-reversal symmetry of the extensive density \( s \) is even. Because of the time-derivative, the intrinsic time-reversal signatures of both the flux \( j^s_i \) and the source \( R/T \) are opposite the intrinsic time-reversal symmetry of \( s \), making them even under \( T \).

Throughout this work, Roman indices indicate real space, and the vector symbol indicate spin space. Moreover, although the present work has solids in mind, it neglects the effects of stress and pressure.

### III. THREE THERMODYNAMIC SYSTEMS

We begin by considering the insulator, a simple but commonly encountered system for which the only thermodynamic force is the temperature gradient, and there is only one thermodynamic flux and one thermodynamic source. We then successively add additional degrees of freedom, which leads to new forces, fluxes, and sources, and off-diagonal Onsager coefficients connecting fluxes and/or sources associated with a given intensive thermodynamic quantity with another thermodynamic force.

#### A. Insulator

For an insulator, in the absence of the effects of stress, the only other equations needed besides that for the entropy are those for the thermodynamics, here given by

\[
d\varepsilon = T \, ds,
\]

and that for the energy density, given by

\[
\partial_i \varepsilon + \partial_t j^s_i = 0,
\]

Here \( j^s_i \) is the energy flux density. There is no energy source, because energy is conserved. The intrinsic time-reversal signature of \( j^s_i \) is odd. We combine (3), (4), and (5) to obtain

\[
0 \leq R = T \partial_i s + T \partial_0 j^s_i = \partial_i \varepsilon + T \partial_t j^s_i = -\partial_i j^s_i + T \partial_t j^s_i
= -\partial_i (j^s_i - T j^s_i) - j^s_i \partial_t T.
\]

To satisfy this equation with \( R \geq 0 \), we eliminate the divergence (which, if non-zero, can have either sign) by setting

\[
j^s_i = T j^s_i,
\]

although \( j^s_i \) is not yet specified.

We now apply the fundamental idea behind irreversible thermodynamics: that for small amplitude deviations from equilibrium the fluxes and sources are linearly proportional to the forces. In the present case, there is only one flux (\( j^s_i \)) and one force \( (\partial_t T) \), so that

\[
\dot{j}^s_i = -\frac{\kappa}{T} \partial_t T,
\]

where it is conventional to call \( \kappa \) the thermal conductivity. This yields

\[
R = \frac{\kappa}{T} (\partial_t T)^2.
\]

To ensure that this is non-negative we require that \( \kappa \geq 0 \). In fact, to attain thermal equilibrium we must have \( \kappa > 0 \). This corresponds to the well-known condition that, for thermal equilibration, heat flows from hot to cold.

#### B. Insulating Paramagnet

For insulating paramagnets (and for ferromagnets), the thermodynamics is given by

\[
d\varepsilon = T \, ds - \mu_0 \vec{H} \cdot d\vec{M},
\]

where both \( \vec{M} \) and the net effective field \( \vec{H} \) have odd intrinsic time-reversal signature. Here we measure \( \vec{H} \) in SI units of A-turn/m, and \( \vec{M} \) in A/m. Since \( \mu_0 = 4\pi \times 10^{-7} \text{T-m/A} \), this means that \( \mu_0 \vec{H} \) has units of tesla.

For paramagnets (but not for ferromagnets) we take

\[
\vec{H} = \vec{H}_0 - \frac{\vec{M}}{\chi},
\]

where \( \vec{H}_0 \) is the applied field and \( \chi \) is the low-field susceptibility. Ref. 35 employs a very similar form. When the energy density \( \varepsilon \) is minimized with respect to \( \vec{M} \), so \( \mu_0 \vec{H} = -\partial \varepsilon/\partial \vec{M} = \vec{0} \), (11) yields \( \vec{M} = \chi \vec{H}_0 \), as expected.

The equation of motion for the magnetization is given by

\[
\partial_t \vec{M} + \partial_i \vec{Q}_i = -\gamma \mu_0 \vec{M} \times \vec{H} + \vec{N}.
\]

Here the gyromagnetic ratio is taken to be \( -\gamma \), where \( \gamma > 0 \). Both the spin (magnetization) flux \( \vec{Q}_i \) and the spin (magnetization) source (or torque density) \( \vec{N} \) have even intrinsic time-reversal signature. It is zero in systems that are rotationally invariant in spin-space, but can be non-zero due to the spin-orbit interaction or the dipole-dipole interaction. Note that (11) implies that \( \vec{M} \times \vec{H} = \vec{M} \times \vec{H}_0 \) in (12).

Combining the dynamical equations for \( \varepsilon, s, \) and \( \vec{M} \) gives

\[
0 \leq R = T \partial_i s + T \partial_0 j^s_i = \partial_i \varepsilon + T \partial_t j^s_i + \mu_0 \vec{H} \cdot \partial_i \vec{M}
= -\partial_i j^s_i + T \partial_t j^s_i + \mu_0 \vec{H} \cdot \partial_i \vec{M}
= -\partial_i (j^s_i - T j^s_i) + \mu_0 \vec{H} \cdot \partial_i \vec{Q}_i
= -j^s_i \partial_i T + \mu_0 \vec{N} \cdot \vec{H} + \mu_0 \vec{Q}_i \cdot \partial_i \vec{H}.
\]

To satisfy this equation with \( R \geq 0 \), we eliminate the divergence (which, if non-zero, can have either sign), by setting

\[
j^s_i = T j^s_i - \mu_0 \vec{H} \cdot \vec{Q}_i.
\]
The other fluxes and sources are given by

\[ j_i^s = -\frac{\kappa}{T} \partial_i T - L_{sQ} \mu_0 \vec{M} \cdot \partial_i \vec{H}, \quad (15) \]
\[ \vec{N} = A \vec{H}, \quad (16) \]
\[ \vec{Q}_i = C \partial_i \vec{H} + C' \vec{M} \times \partial_i \vec{H} - L_{Qs} \vec{M} \partial_i T. \quad (17) \]

Here \( L_{sQ} \) and \( L_{Qs} \) are off-diagonal Onsager coefficients. The time-reversal properties of the associated terms determine that they are both dissipative. Likewise, \( A \) and \( C \) are dissipative, but \( C' \) is reactive.

Note the following: (a) In (15), the term in \( \vec{M} \cdot \partial_i \vec{H} \), with coefficient \(-L_{sQ}\), appears to be new, although it is small because the system is assumed to have only a small magnetization relative to the saturation value \( M_s \). (b) The term \( A \vec{H} \) in (16) corresponds to \( T_1 \) processes in a nuclear spin system, since it gives a term in \( \partial_i \vec{M} \) of \( A(\vec{H}_0 - \vec{M}/\chi) \). Thus \( A/\chi \) is the same as \( T_1^{-1} \). Since \( \chi \) is dimensionless, \( A \) has units of \( s^{-1} \). Here \( \vec{H} \) serves as a thermodynamic force. (c) In (17) the \( \partial_i \vec{H} \) term corresponds to diffusion, since for uniform \( \vec{H}_0 \) it gives a term in \( \partial_i \vec{M} \) of \(-C \nabla^2 (\vec{H}_0 - \vec{M}/\chi) = (C/\chi) \nabla^2 \vec{M} \). Thus \( C/\chi \) is the diffusion coefficient \( D \), whose units are [velocity-length], where the length corresponds to an appropriate mean-free-path. Here \( \partial_i \vec{H} \) serves as a thermodynamic force. The \( \vec{M} \times \partial_i \vec{H} \) term in (17) is reactive, and corresponds to an induced exchange field, since as noted by Leggett in his study of paramagnetic \(^3\)He, the \( \vec{M} \partial_i T \) term in (17), proportional to \( L_{Qs} \), appears to be new. Both \( L_{Qs} \) and \( L_{sQ} \) have units of [velocity-length/temperature]. (d) Eq. (16) omits a symmetry-allowed \( \vec{M} \times \vec{H} \) term, thus assuming that the Zeeman term in (12) exhausts terms of that symmetry. (e) Since no real-space vector \( v_i \) characterizes the equilibrium state, \( \vec{Q}_i \) contains no term in \( v_i \vec{M} \cdot \vec{H} \).

Using these results, (13) becomes

\[ 0 \leq R = \frac{\kappa}{T} (\partial_i T)^2 + A \mu_0 (\vec{H})^2 + C \mu_0 (\partial_i \vec{H})^2 \]
\[ - (L_{sQ} + L_{Qs}) \mu_0 \partial_i T (\vec{M} \cdot \partial_i \vec{H}). \quad (18) \]

With \( \kappa, A, \) and \( C \) all non-negative, this ensures that \( R \) is non-negative. Like \( R, \partial_i T(\vec{M} \cdot \partial_i \vec{H}) \) has positive time-reversal symmetry, so this term is allowed to contribute to \( R \). Then, according to Onsager’s reciprocity principle, the associated off-diagonal transport coefficients satisfy \( L_{sQ} = L_{Qs} \), and can be of either sign. These terms cannot be of too large a magnitude, or \( R \) could become negative, signifying an instability whereby dissipation cools down the system. For thermal stability \( L_{sQ}^2 \leq \kappa \mu_0 C/T \).

C. Insulating Ferromagnet

For a ferromagnet in a minor loop where \( \vec{M} = M_0 \vec{H}_0 + \chi \vec{H}_0 \) in equilibrium, we include a non-uniform exchange term \( A'(\partial_i \vec{M})^2 \) in the energy density. Thus we take

\[ \mu_0 \vec{H} = -\frac{\delta \varepsilon}{\delta \vec{M}} = -\partial_i \vec{H} + \partial_i (\partial_i \vec{M}) \]
\[ = \mu_0 \vec{H}_0 + 2A' \nabla^2 \vec{M} - \mu_0 \vec{M} - M_0 \vec{M}. \]

In the absence of non-uniform exchange, Ref. 35 employs a very similar form. If the energy is minimized, so \( \vec{H} = \vec{0} \), then in the absence of non-uniform exchange this yields \( \vec{M} = M_0 \vec{H}_0 + \chi \vec{H}_0 \), as desired. Hence the magnitude \( |\vec{M}| \) is permitted to vary. An anisotropy field \( \vec{H}_{an} \) may also be relevant, but is omitted. The uniform part of the exchange field, which is along \( \vec{M} \), does not contribute to \( \vec{M} \times \vec{H} \).

In addition, \( A \) and \( C \) of (16) and (17) must be replaced by tensors determined by \( \vec{M} \). Thus we now employ

\[ A_{\alpha \beta} = (A_{\parallel} - A_{\perp}) \vec{M}_\alpha \vec{M}_\beta + A_{\perp} \delta_{\alpha \beta}, \quad (20) \]

where \( \delta_{\alpha \beta} \) is the Kronecker delta in spin space; and similarly for \( C \).

With these forms and the identity that, for an arbitrary vector \( \vec{G} \), the part transverse to \( \vec{M} \) is given by \( \vec{G}_\perp = \vec{G} - \vec{M} \times (\vec{M} \times \vec{G}) \), we have

\[ \vec{N} = A_{\parallel} (\vec{M} \cdot \vec{H}) \vec{M} - A_{\perp} \vec{M} \times (\vec{M} \times \vec{H}), \]
\[ \vec{Q}_i = C_{\parallel} (\vec{M} \cdot \partial_i \vec{H}) \vec{M} - C_{\perp} \vec{M} \times \vec{H} - L_{Qs} \vec{M} \partial_i T. \]

(21) (22)

\( A_{\parallel}/\chi \) is the same at \( T_1^{-1} \), and describes longitudinal (Bloch) relaxation. Moreover, if we set

\[ A_{\perp} = \lambda \mu_0 M, \]

(23)

where \( \lambda \) is the Landau-Lifshitz damping parameter (in \( (T m)^{-1} \)) and \( M = |\vec{M}| \), then the transverse part of \( \vec{N} \) is precisely the Landau-Lifshitz form. As for the paramagnet, (16) omits a symmetry-allowed \( \vec{M} \times \vec{H} \) term, thus assuming that the explicit torque term in (12) exhausts terms of that symmetry.

Using these results, (13) becomes

\[ 0 \leq R = \frac{\kappa}{T} (\partial_i T)^2 + \mu_0 A_{\parallel} (\vec{M} \cdot \vec{H})^2 + \mu_0 A_{\perp} (\vec{M} \times \vec{H})^2 \]
\[ + \mu_0 C_{\parallel} (\vec{M} \cdot \partial_i \vec{H})^2 + \mu_0 C_{\perp} (\vec{M} \times \partial_i \vec{H})^2 \]
\[ - (L_{sQ} + L_{Qs}) \partial_i T (\vec{M} \cdot \partial_i \vec{H}). \]

(24)

Clearly all of the coefficients \( \kappa, A_{\parallel}, A_{\perp}, C_{\parallel}, \) and \( C_{\perp} \) must be non-negative. \( L_{sQ} = L_{Qs} \), and can have either sign, as discussed above for the paramagnet. For thermal stability \( L_{sQ}^2 \leq \kappa \mu_0 C/T \).

IV. LANDAU-LIFSHITZ VERSUS GILBERT DAMPING

In the absence of spin flux \( \vec{Q}_s \), the equation of motion for a uniform ferromagnet, (12), takes the form

\[ \partial_t \vec{M} = -\gamma \mu_0 \vec{M} \times \vec{H} + \vec{N}. \]

(25)
For fixed $|\vec{M}|$, we have $A_{||} = 0$, so $\vec{N} = -\lambda \mu_0 \vec{M} \times (\vec{M} \times \vec{H})$, which yields the Landau-Lifshitz form of damping.

On the other hand, Gilbert argues, by analogy to damping of a particle using the Rayleigh damping term in a modified Hamiltonian formulation of mechanics, that the damping form should go as $\vec{M} \times \partial_t \vec{M}$.

However, $\partial_t \vec{M}$ is not a thermodynamic force, does not have a unique time-reversal signature, and furthermore gives the equation of motion a curious self-referential character:

$$\partial_t \vec{M} = -\gamma G \mu_0 \vec{M} \times \vec{H} + \alpha \vec{M} \times \partial_t \vec{M} \ (\text{Gilbert}) \ (26)$$

On eliminating this self-referential aspect of Gilbert damping, the Gilbert equation can be put into the same form as the Landau-Lifshitz equation (25). A consequence is that the Landau-Lifshitz (true) gyromagnetic ratio $\gamma$ takes on a smaller value than $\gamma_G = \gamma (1 + \alpha^2)$; damping “renormalizes” the Zeeman interaction! We find it unusual that a fundamental interaction should be renormalized by damping, and always in the same fashion: no matter the source of the damping, if it gives the same $\alpha$, then it gives the same renormalized $\gamma_G$. (Also, $\alpha = \lambda / \gamma$.) It strikes us as unlikely that a microscopic theory that includes all sorts of terms capable of producing damping (terms that can even interfere with one another) should yield any renormalization of $\gamma$ at all, no less in such a specific and interaction-independent fashion. (Properly, renormalization by such terms should be calculated directly, as an equilibrium properly, and distinct from damping.) As noted in the Introduction, we consider this to be a major conceptual difficulty with Gilbert damping, in addition to its lack of concord with irreversible thermodynamics. It may be capable of being tested, given that $\alpha \approx 0.8$ systems occur.

As an aside we note that one argument sometimes given in favor of Gilbert theory is that it gives critical damping, as for a damped harmonic oscillator. However, the harmonic oscillator has a resonant frequency due to mass (it goes to infinity as the mass goes to zero), so that when one stops driving a harmonic oscillator it continues to move. Spins, however, do not have mass, and stop instantaneously when the net field $\vec{H}$ goes to zero, with damping terms of either LL or Gilbert form.

Any analogy of Gilbert theory to the harmonic oscillator must address this issue of inertia. Moreover, for the harmonic oscillator the damping force is opposite the rate of change of the velocity; for Gilbert theory the damping torque is perpendicular to the rate of change of the angular momentum. The mathematical structures of the two systems, spin and harmonic oscillator, are fundamentally different; the physical mechanism whereby Gilbert theory can give critical damping of spins is totally different than for the harmonic oscillator.

Irreversible thermodynamics does have limitations. For large-angle motions the losses of a given driven mode increase because of (among other possible things) the onset of spin wave instability processes. These are beyond the purview of irreversible thermodynamics.

The choice of Landau-Lifshitz or Gilbert damping does not affect microscopic calculations, unless they assume a non-thermal distribution based on Gilbert damping. For microscopic calculations unbiased relative to LL or G, only the interpretation is affected. Note that a recent evaluation for itinerant magnets finds good agreement for Fe, Co, and Ni. To close this section, we note that Ref.41 obtains a Langevin theory for the Gilbert damping constant, but using a thermodynamic distribution that builds in Gilbert damping, rather than using an energy-based thermodynamic distribution.

V. ANALOGY BETWEEN MAGNETS AND FLUIDS IN POROUS MEDIA

It is not uncommon to read of an analogy between magnets and fluids, with reference to magnetic hysteresis as evidence for magnetic “viscosity”. Although magneticians certainly know what they mean by this, the analogy to fluids is a bit more complex than one might imagine, and we take the opportunity here to clarify this point.

In fluids, viscosity is a measure of diffusion of linear momentum. The proper magnetic analog is the magnetic diffusion constant. What, then, is the fluid analog of magnetic damping? In the case of magnetic damping, angular momentum is transferred to the lattice. For a fluid with a large open volume, there is nothing to which the fluid can transfer linear momentum. Therefore a better analogy of a magnet would be to a fluid that fills a porous medium; preferably a fluid that is not too viscous, so that motions analogous to magnetic oscillations are not immediately damped out.

However, this analogy remains imperfect, because the fluid has inertial mass whereas the magnet has no angular inertia (i.e. no moment of inertia).

VI. NON-UNIFORM INSULATING FERROMAGNET

This system’s thermodynamics is described by (10), its constitutive relation by (19) and its dynamical equations are (5), (3), and (12). However, now the structure of the equilibrium system is characterized by both $\vec{M}$ and $\partial_t \vec{M}$, which permits additional terms in the fluxes and sources.

The energy flux is still given by (14), but now the entropy flux has three additional terms. Explicitly,

$$j_i^s = \frac{k}{T} \partial_i T - L_s \mu_0 \vec{M} \cdot \partial_i \vec{H} - L_{sM1} \mu_0 (\partial_i \vec{M}) \cdot \vec{H} - L_{sM2} \mu_0 (\vec{M} \cdot \partial_i \vec{M}) \cdot \vec{H} \ (27)$$

Thus, if there is a torque and a texture (non-zero $\partial_i \vec{M}$), the entropy flux (and the energy flux) can be driven by “pumping” on the magnetization via $\vec{M} \times \vec{H}$. Here the
coefficients $L_{sM}$ and $L_{sM2}$ (both dissipative), and $L'_{sM}$ (reactive) all have units of [velocity-length/temperature]. (The time-reversal properties of these terms determine their dissipative or reactive nature.) For a uniform magnetization $\mathbf{M} \cdot \partial_t \mathbf{M} = 0$, in which case the term $L_{sM2}$ is irrelevant.

The spin flux, whose conjugate force is $\partial_t \mathbf{H}$, has eight new terms, all of them associated with the thermodynamic force $\mathbf{H}$. Explicitly,

$$
\tilde{Q}_{i} = C_{\|} \mathbf{M} \cdot \partial_t \mathbf{H} - L_{Q} \mathbf{M} \times \partial_t \mathbf{H}
$$

$$
- C' \mathbf{M} \times \partial_t \mathbf{H} + L_{QN1} \partial_t \mathbf{M} \mathbf{H} + L_{QN2} (\mathbf{M} \cdot \partial_t \mathbf{M}) \mathbf{H}
$$

$$
+ L_{QN3} \mathbf{M} \partial_t \mathbf{M} \mathbf{H} + L_{QN4} \mathbf{M} \partial_t \mathbf{M} \mathbf{H} - L_{QN1} \mathbf{M} \partial_t \mathbf{M} \mathbf{H} + L_{QN2} \mathbf{M} \partial_t \mathbf{M} \mathbf{H}
$$

$$
+ L'_{QN3} \mathbf{M} \times \partial_t \mathbf{M} \mathbf{H} + L'_{QN4} \mathbf{M} \times \partial_t \mathbf{M} \mathbf{H}.
$$

(28)

The Onsager terms without (with) primes are dissipative (reactive). For a uniform magnetization $\mathbf{M} \cdot \partial_t \mathbf{M} = 0$, in which case the terms in $L_{QN2}$, and $L_{QN4}$ are irrelevant. The $L_{QN}$ terms all have units of [velocity-length/magnetization].

The spin source, or spin torque, whose conjugate force is $\mathbf{H}$, has eight new terms, three associated with $\partial_t T$ and eight associated with $\partial_t \mathbf{H}$. Explicitly,

$$
\tilde{N} = A_{\|} \partial_t \mathbf{H} - A_{\perp} \mathbf{M} \times \partial_t \mathbf{H}
$$

$$
+ L_{MN1} \partial_t \mathbf{M} \partial_t T + L_{MN2} \mathbf{M} \partial_t \mathbf{M} \partial_t T + L_{MN3} \mathbf{M} \partial_t \mathbf{M} \partial_t T
$$

$$
+ L_{MNQ1} \partial_t \mathbf{M} \partial_t \mathbf{H} + L_{MNQ2} \mathbf{M} \partial_t \mathbf{M} \partial_t \mathbf{H}
$$

$$
+ L_{MNQ3} \mathbf{M} \partial_t \mathbf{M} \partial_t \mathbf{H} + L_{MNQ4} \mathbf{M} \partial_t \mathbf{M} \partial_t \mathbf{H}
$$

$$
+ L'_{MNQ1} \mathbf{M} \partial_t \mathbf{M} \partial_t \mathbf{H} + L'_{MNQ2} \mathbf{M} \partial_t \mathbf{M} \partial_t \mathbf{H}
$$

$$
+ L'_{MNQ3} \mathbf{M} \partial_t \mathbf{M} \partial_t \mathbf{H} + L'_{MNQ4} \mathbf{M} \partial_t \mathbf{M} \partial_t \mathbf{H}.
$$

(29)

The Onsager terms without (with) primes are dissipative (reactive). For a uniform magnetization $\mathbf{M} \cdot \partial_t \mathbf{M} = 0$, in which case the terms in $L_{MN2}$, $L_{MNQ2}$, $L_{MNQ4}$ and $L_{MNQ4}$ disappear. The $L_{MNQ}$ terms all have units of [velocity-length/magnetization].

In principle, the flux and torque terms involving new Onsager coefficients can contribute for non-uniform insulating ferromagnets, especially when the texture is varying rapidly in space. Of course, values for the new Onsager coefficients are not known.

Since $R$ must be invariant under time-reversal, when the above equations are substituted to find $R$ we must eliminate the cross-terms that are odd under $T$. This leads to the constraints that

$$
L'_{MN} = -L'_{M}, \quad L'_{MNQ1} = +L'_{QN1}, \quad L'_{MNQ2} = -L'_{QN2}, \quad L'_{MNQ3} = -L'_{QN3}, \quad L'_{MNQ4} = +L'_{QN4}.
$$

(30)

In addition, to satisfy the Onsager relations the contributions to $R$ from cross-terms are equal. Thus we must take

$$
L_{sQ} = L_{Q5}, \quad L_{sM1} = L_{M31}, \quad L_{sM2} = L_{M32}, \quad L_{NQ1} = L_{QN1}, \quad L_{NQ2} = L_{QN2}, \quad L_{NQ3} = L_{QN3}, \quad L_{NQ4} = L_{QN4}.
$$

(31)

With these forms (13) becomes

$$
R = \frac{\kappa}{T} (\partial_t T)^2 + \mu_0 A_{\|} (\mathbf{M} \cdot \partial_t \mathbf{H})^2 + \mu_0 A_{\perp} (\mathbf{M} \times \partial_t \mathbf{H})^2 + 2L_{sQ}\mu_0 (\mathbf{M} \cdot \partial_t \mathbf{H}) \partial_t T
$$

$$
+ 2L_{sM1}\mu_0 (\mathbf{M} \cdot \partial_t \mathbf{M}) \partial_t T
$$

$$
+ 2L_{sM2}\mu_0 (\mathbf{M} \cdot \partial_t \mathbf{M})(\mathbf{M} \cdot \partial_t \mathbf{H}) \partial_t T
$$

$$
+ 2\mu_0 L_{NQ1} \partial_t \mathbf{M} \partial_t \mathbf{H} (\mathbf{M} \cdot \partial_t \mathbf{H})
$$

$$
+ 2\mu_0 L_{NQ2} (\mathbf{M} \cdot \partial_t \mathbf{M})(\mathbf{H} \cdot \partial_t \mathbf{H})
$$

$$
+ 2\mu_0 L_{NQ3} (\partial_t \mathbf{M} \mathbf{H} \cdot \mathbf{M} \cdot \partial_t \mathbf{H}).
$$

(32)

VII. DAMPING AND NORMAL MODES

Damping can be treated as a perturbation when applied to the normal modes, so the effect of the primed transport coefficients (reactive) can be neglected. Moreover, because simulations consider only the transverse degrees of freedom (so $\mathbf{M} \cdot \partial_t \mathbf{M} = 0$) and constant temperature, the relevant part of the transverse part of the equation of motion for $\mathbf{M}$ is all that we consider now. In addition to the Larmor term and the Landau-Lifshitz term, we must include the $C_{\|}, C_{\perp}, L_{QN1}, L_{QN3}, L_{QN1}$, and $L_{QN3}$ terms. Most of the additional terms have the form $(\partial_t \mathbf{M}) \mathbf{H}$. These are non-zero because motions of neighboring spins can align with the equilibrium spin orientation. There are also terms of the form $\mathbf{M} \times (\mathbf{M} \times \mathbf{H}^2) = -(\mathbf{H}^2 \mathbf{M}) \mathbf{H}$ and $(\mathbf{H}^2 \mathbf{M}) \mathbf{H}$. Neglecting the reactive corrections and terms that are along $\mathbf{M}$, we have

$$
\partial_t \mathbf{M} \mathbf{H} \approx -\gamma \mu_0 \mathbf{M} \mathbf{H} + (\nabla \mathbf{M}) \mathbf{H}
$$

$$
- \gamma \mu_0 \mathbf{M} \mathbf{H} + \mathbf{M} \mathbf{H} \mathbf{M} - C_{\parallel} \mathbf{H}^2 \mathbf{H}
$$

$$
- (C_{\perp} - C_{\parallel}) (\partial_t \mathbf{M}) (\mathbf{M} \cdot \partial_t \mathbf{H})
$$

$$
+ (L_{NQ3} - L_{QN1}) (\partial_t \mathbf{M}) (\mathbf{M} \cdot \partial_t \mathbf{H})
$$

$$
- L_{NQ1} (\mathbf{H}^2 \mathbf{M}) \mathbf{H}
$$

$$
- (L_{NQ1} + L_{NQ3}) (\partial_t \mathbf{M}) (\mathbf{M} \cdot \partial_t \mathbf{H}).
$$

(33)

When implemented in a magnetics code, the derivatives become differences. Note that quantities like $(\partial_t \mathbf{M}) \mathbf{H}$, $(\mathbf{H}^2 \mathbf{M}) \mathbf{H}$, and $\mathbf{M}$ are evaluated in equilibrium; only $\mathbf{H}$ and $\partial_t \mathbf{H}$ are non-equilibrium quantities.

In principle, one can derive expressions for the new non-uniform Onsager coefficients in terms of various correlation functions of the system, and then evaluate these expressions. However, in light of how long it has taken to do the same thing for uniform (conducting) ferromagnets, a more likely approach is as follows:
(1) identify data where there seem to be size effects associated with the damping and unrelated to conduction processes, \(^\text{(45)}\); and not associated with surface effects; \(^\text{(46)}\) perform simulations with trial values of the new Onsager coefficients. By a process of trial-and-error it should be possible to determine the most important terms and to estimate their values, and then refine them as data accumulates.

VIII. SUMMARY

We have considered the implications of irreversible thermodynamics for magnetic insulators, including paramagnets, uniform ferromagnets, and nonuniform ferromagnets. Two apparently new terms appear in the off-diagonal transport properties associated with temperature gradients and field gradients. As advertised, we show that irreversible thermodynamics uniquely predicts Landau-Lifshitz damping for a uniform ferromagnet. In addition, the irreversible thermodynamics of non-uniform ferromagnets contains a number of new terms in the entropy flux, spin flux, and spin torque, associated with the fact that \(\partial_i \vec{M} \) is non-zero. These terms may be of significance for small systems, where boundary conditions constrain the equilibrium magnetization to vary rapidly in space.

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