Don't waste time on problems you aren't sure of. Be clear and concise. A cluttered response will not get full credit.

1. (8 pts) Let $E_x = 4x - 2$, with $E_x$ in volts/m and $x$ in m. If $V = 0$ at $x = 0$, find $V(x)$. Plot $E_x$ and $V(x)$ in the $x$-interval $(-1 \text{ m}, 1 \text{ m})$.

\[
V(x) - V(0) = \int_0^x E_x \, dx = -\int_0^x (4x-2) \, dx = -\left[2x^2 - 2x\right]_0^x = -(2x^2 - 2x)
\]

\[
V(x) = V(0) - 2x^2 + 2x = -2x^2 + 2x = -2(x - \frac{1}{2})^2 + \frac{1}{2}
\]

2. (8 pts) A charge $q > 0$ is uniformly distributed over a rod of length $a$ placed on the $x$-axis with its right end at the origin. Find the voltage at $x$ on the positive $x$-axis.

\[
dq = \frac{q}{2\pi\lambda} \, dx' = \frac{q}{a} \, dx'
\]

\[
\int V = \frac{kq}{x-x'} = \frac{kq}{a} \frac{dx'}{x-x'}
\]

Let $r = x - x'$, so $dr = -dx'$. Then,

\[
\int V = -\frac{kq}{a} \int_0^r \frac{dr}{x+a} = -\frac{kq}{a} \ln (\frac{x+z}{x})
\]

3. (8 pts) A fixed point charge $Q$ is at the origin. At $t = 0$ a charge $q$ with mass $m$ is at $x = a$ with leftward velocity $v_0$ that satisfies $kQq/a = 3mv_0^2$. (a) Find $b/a$, where $b < a$ is the position where $q$ turns around and starts to move rightward. (b) Find the velocity $v_\infty$ of $q$ at large distances from the origin, in the form $v_\infty = \ldots$.

\[
(2) \quad \frac{1}{2} m v_0^2 + \frac{kqQ}{a} = \text{cons}, \quad \text{Energy} \quad E \quad V = 0 \quad \text{at} \quad r = b.
\]

Thus, $\frac{1}{2} \frac{kqQ}{a} = \frac{kqQ}{b}$, so $b = \frac{b}{a}$.

\[
(1) \quad \frac{1}{2} m v_0^2 + \frac{kqQ}{a} = \frac{1}{2} m v_\infty^2 + Q \frac{kqQ}{r} \rightarrow 0 \quad \text{as} \quad r \rightarrow a.
\]

Thus, $\frac{1}{2} m v_0^2 = \frac{1}{2} m v_\infty^2$, so $v_\infty = \sqrt{7} v_0$. 


4. (8 pts) Derive the capacitance of a sphere of radius $R$. Explain your reasoning.

\[ C = \frac{Q}{\Delta V} = \frac{Q}{V - V_{\infty}} \]

For a point charge and for a spherical charge, \( \frac{V^2}{R} = \frac{kQ}{r} \)
outside, so \( V = \frac{kQ}{r} + V_{\infty} \) in both cases.
Hence, with \( V_{\infty} = 0 \) at \( R \) we have \( V = \frac{kQ}{R} \).

Thus \( C = \frac{Q}{kV} = \frac{R}{k} \).

5. Consider three capacitors. \( C_1 = 10 \mu F \) and \( C_2 = 20 \mu F \) are in parallel, and \( C_3 = 20 \mu F \) is in series with them. \( V_A = 10 \) V and \( V_C = -5 \) V.
a. (8 pts) Find the charge and voltage difference for each capacitor. Find \( V_B \).

\[ \Delta V = V_A - V_C = 15 \text{ V} \]

\[ C_{\text{eff}} = \left( C_3^{-1} + \left( C_1 + C_2 \right)^{-1} \right)^{-1} = \left( \frac{1}{20} + \left( \frac{1}{30} \right)^{-1} \right)^{-1} = \left( \frac{2}{30} + \frac{1}{20} \right)^{-1} = 12 \mu F \]

\[ Q = Q_3 = C_{\text{eff}} \cdot \Delta V = 180 \mu C \]

\[ \Delta V_3 = \frac{Q_2}{C_3} = \frac{180 \mu C}{20 \mu F} = 9 \text{ V} \]

so \( V_B = V_A - \Delta V_3 = 10 - 9 = 1 \text{ V} \)

\[ \Delta V_1 = \Delta V_2 = V_B - V_C = 6 (-5) = 6 \text{ V} \]

so \( Q_1 = C_1 \Delta V_1 = 10 \times 6 = 60 \mu C \)

\( Q_2 = C_2 \Delta V_2 = 20 \times 6 = 120 \mu C \)

b. (8 pts) If, using insulating gloves, \( C_3 \) is disconnected and then placed in parallel with \( C_1 \) and \( C_2 \), find the new charge and voltage difference for each capacitor.

\[ Q = Q_1 + Q_2 + Q_3 = 60 + 120 + 180 = 360 \mu C \]

\[ C_{\text{eff}} = C_1 + C_2 + C_3 = 10 + 20 + 20 = 50 \mu F \]

\[ \Delta V_1' = \Delta V_2' = \Delta V_3' = \frac{Q}{C_{\text{eff}}'} = \frac{360}{50} = 7.2 \text{ V} \]

\[ Q_1' = C_1 \Delta V_1' = 10 \times (7.2) = 72 \mu C \]

\[ Q_2' = C_2 \Delta V_2' = 20 \times (7.2) = 144 \mu C \]

\[ Q_3' = C_3 \Delta V_3' = 20 \times (7.2) = 144 \mu C \]

\[ Q_1' + Q_2' + Q_3' = 360 \mu C \]
6. A parallel plate capacitor has electrical energy $7.2 \times 10^{-5}$ ergs when connected to a 6 V battery. It is now disconnected from the battery. A slab of dielectric constant $\kappa = 4$ and nearly the same thickness as the capacitor is slid into the capacitor.

a. (2 pts) What is the voltage difference now?

$$\Delta V = \frac{\Delta V_0}{\kappa} = \frac{6}{4} = 1.5 \text{ V}$$

b. (2 pts) What is the electrical energy now?

$$U = \frac{1}{2} C_0 (\Delta V_0^2) = \frac{1}{2} (\kappa C_0) \left( \frac{\Delta V_0}{\kappa} \right)^2 = \frac{1}{2} C_0 \left( \frac{6}{4} \right)^2 = \frac{U_0}{\kappa} = 1.6 \times 10^{-5} \text{ ergs}$$

c. (4 pts) Was the dielectric attracted, repelled, or did it feel no force when it was part way in the capacitor, and why? (No reason, no credit.)

Attracted, since the final energy is lower.
(It gets polarized and is attracted by the induced field.)

7. (8 pts) You are given a voltaic cell with internal resistance of 4 $\Omega$. When shorted, it briefly produces a current of 0.15 A.

a. (4 pts) Find its emf and the rate at which it energy discharged.

$$I_{\text{short}} = \frac{E}{r} \Rightarrow E = I_{\text{short}} \cdot (r) = 0.15 \times 4 = 0.6 \text{ V}$$

$$P = I_{\text{short}}^2 \cdot r = (0.15^2 \frac{4}{4}) = 0.09 \text{ W}$$

b. (4 pts) There is a certain load resistance for which this voltaic cell will provide maximum power to load. Find that resistance and that maximum power.

$$R_L = r \text{ (impedance matching)}$$

$$R_L = 4 \Omega \Rightarrow P = I^2 R_L = \left( \frac{E}{r + R_L} \right)^2 R_L = \left( \frac{0.6}{2} \right)^2 = \frac{E^2}{4r_L} = \frac{0.36}{16} \Rightarrow P = 0.23 \text{ W}$$

c. (4 pts) A 100% efficient flashlight bulb produces 0.9 W when used with the AA cell. If its resistance $R$ is much larger than the cell's internal resistance, find $R$ and the efficiency at which the battery produces useful power.

Impossible problem! Bad choice of numbers. Can't produce more power less than under shorting.

$$P_{\text{bulb}} = 0.9 \text{ W} \text{ (very low)}, \text{ than} \frac{E^2}{4r_L} R_L \ll \frac{E^2}{r}$$

$$R_L \approx \frac{E^2}{P} = \frac{0.36}{0.9} = 0.4 \Omega \Rightarrow r = 4 \Omega$$

$$\text{Efficiency} = \frac{R_L}{R + R_L} = \frac{4}{4 + 4} = 83\%$$
8. (6 pts) A sluiceway has length 80 m and cross-section 5 m². Fish move through it with average velocity 0.4 m/s, with each taking up a volume of 2 m³. Find the rate at which the fish pass through the exit. If they each carry a charge of $10^{-6} \text{ C}$, find the electric current passing through the exit.

\[
\frac{dN}{dt} = \frac{N}{t} = \frac{n(A)A}{t} = \frac{16/6}{(2\text{ m}^3)(5\text{ m}^2)(0.4\text{ m}/s)} = \frac{1\text{ fish}}{\text{sec}}
\]

\[
\frac{dQ}{dt} = Q \frac{dN}{dt} = 10^{-6} \frac{\text{C}}{\text{sec}} = 10^6 \text{A} = 1\mu\text{A}
\]

9. A voltaic cell has internal resistance $r = 0.25 \Omega$ and open circuit voltages across the left and right electrodes of 0.2 V and 1.4 V, for a net emf of $E = 1.2 \text{ V}$. It is in series with a resistor $R = 0.55 \Omega$. Let $V_a = 0.3 \text{ V}$. The connecting wires have zero resistance.

a. (8 pts) Find the current, the voltage drops across the resistances, and sketch the voltage around the circuit.

\[
\frac{E}{R+r} = \frac{1.2}{1.55} = 0.78 \text{ A}
\]

\[
IR = (1.5)(0.35) = 0.825 \text{ V}
\]

\[
IR = (1.5)(0.25) = 0.375 \text{ V}
\]

\[
IR = 0.825
\]

b. (2 pts) If the voltaic cell discharges in 50 minutes, find its initial "charge" and its initial energy.

10. (8 pts) Find the unknown currents, the unknown resistance, and the unknown emf for the circuit in the figure.

\[
(6A)(6\Omega) = \sum R = 4R
\]

\[
\Rightarrow R = \frac{3\Omega}{4} = 0.75 \Omega
\]

To get $E$, start at the ground and go counterclockwise until you get to $E$.

The voltage rises by $3 \times 5 + 4 \times 10 + 0 + 6 + 6\times6 = 15 + 40 + 0 + 6 + 6\times6 = 97 \text{ V}$

Starting at ground and going clockwise gives $-2/10 + E = -20 + E$.
11. (12 pts) For the circuit below, take $E_1 = 8 V$, $E_2 = 11 V$, $E_3 = 14 V$, $r_1 = 0.01 \Omega$, $r_2 = 0.04 \Omega$, $r_3 = 0.02 \Omega$, $R = 0.03 \Omega$. (1) Indicate and label the directions of positive currents and indicate the positive side of the voltage $\Delta V$ across $R$. (2) Analyze the circuit using Kirchhoff’s rules. (3) Solve for the voltage across $R$. (4) Find the current through $R$ and the currents provided by each of the batteries.

\[ \begin{align*}
\Delta V &= E_1 - E_2 - E_3/R_1 + 1/R_1 + 1/R_2 + 1/R_3 \\
800 + 700 - 275 &= \Delta V \left( \frac{1000 + 33.3 + 100 + 25 + 50}{208.3} \right) \\
122.5 &= \Delta V \\
\Rightarrow \Delta V &= 5.88 V
\end{align*} \]

\[ \begin{align*}
I &= \frac{\Delta V}{R} = \frac{5.88}{0.3} = 19.6 A \\
I_1 &= \frac{E_1 - \Delta V}{r_1} = \frac{8 - 5.88}{0.1} = 212 A \\
I_2 &= \frac{E_2 - \Delta V}{r_2} = \frac{11.5 - 5.88}{0.04} = 422 A \\
I_3 &= \frac{E_3 - \Delta V}{r_3} = \frac{14.5 - 5.88}{0.02} = 406 A
\end{align*} \]

12. The capacitor is uncharged initially. The switch is then closed at $t = 0$. Let $E = 8 V$, $r = 2 \Omega$, $R_1 = 12 \Omega$, $R_2 = 6 \Omega$, $C_1 = 4 \mu F$.

a. (8 pts) Find $I$, $Q_1$, $I_1$, and $I_2$ just after the switch is closed. Explain.

\[ \begin{align*}
Q_1 &= 0 \text{ at } t = 0, \text{ so } C_1 \text{ has } \Delta Q_1 = 0 \text{ so } \Delta V = 0 \text{ can be neglected.} \\
\text{Then } R_{eq} &= R + \left( R_1' + R_2' \right) = 2 + \left( \frac{1}{12} + \frac{1}{6} \right)^{-1} = 2 + 4 = 6 \Omega \\
\text{Thus } I &= \frac{E}{R_{eq}} = \frac{8}{6} = \frac{4}{3} A
\end{align*} \]

\[ \begin{align*}
\Delta V_1 &= \Delta V_2 = E - R = 8 - \frac{4}{3} \cdot 2 = \frac{16}{3} V \\
I_1 &= \frac{\Delta V_1}{R_1} = \frac{16}{3} \cdot \left( \frac{4}{3} A \right) = \frac{16}{3} \cdot \frac{4}{3} A = \left( \frac{8}{3} A \right) \Rightarrow \left( I_1 + I_2 = \frac{4}{3} A = I \right)
\end{align*} \]

b. (6pts) Find $I$, $Q_1$, $I_1$, and $I_2$ a long time after the switch is closed. Explain.

When $C_1$ has charged up, $\left( I_1 = 0 \right)$.

Then $I = I_2 = \frac{E}{r + R_2} = \frac{8}{2 + 6} = 1 A$

For $Q_1$, $\Delta V_1 = \Delta V_2 = I_2 R_2 = \left( 1 \right) \left( 6 \right) = 6 V$

So $Q_1 = C_1 \Delta V_1 = \left( 4 \mu F \right) \left( 6 V \right) = 24 \mu C$