Don’t waste time on questions you aren’t sure of. Be clear and concise. A cluttered response, some of which is correct and some of which is incorrect, will not get full credit. Five points for your correct section.

1. (15 pts) Let \( E_x = -x^2 + 4 \), with \( E_x \) in volts/m and \( x \) in m. If \( V = -2 \) at \( x = 0 \), find \( V(x) \).
   Evaluate \( E_x \) for \( x = -2, -1, 0, 1, 2 \) m and plot (no calculator plot; if you can’t do this yourself you should ask yourself if you belong in engineering or science).

   \[
   V(x) - V(0) = -\int_0^x E_x \, dx = -\int_0^x (-x^2 + 4) \, dx
   \]
   \[
   = \left[ -\frac{1}{3} x^3 - 4x \right]_0^x = \frac{1}{3} x^3 - 4x.
   \]
   Thus \( V(x) = V(0) + \frac{1}{3} x^3 - 4x \)
   or \( V(x) = -2 + \frac{1}{3} x^3 - 4x \)

2. (20 pts) A spherical capacitor has inner radius \( a \) with charge \( Q \) and outer radius \( b \) with charge \(-Q\), separated by a dielectric with dielectric constant \( \kappa \). Starting from the definition of \( C \) in terms of charge and voltage difference, derive an algebraic expression for the capacitance \( C \).
   (a) Find the radial electric field \( E_r \) in the relevant region of space. Explain your reasoning.

   The system has radial symmetry so \( E = \frac{Q}{\kappa r^2} \).
   By Gauss’s Law applied to a concentric sphere,
   \[
   \oint \mathbf{E} \cdot d\mathbf{A} = \frac{4\pi}{\kappa} Q_{\text{enc}} \]
   gives \( \oint \mathbf{E} \cdot d\mathbf{A} = \oint E_r \, r \, dr = \oint E_r \, r \, dA \),
   so \( \oint E_r \, dA = E_r (4\pi r^2) . \) For \( a < r < b \), we have \( Q_{\text{enc}} = Q \).
   Thus, for \( a < r < b \), Gauss’s Law gives \( E_r (4\pi r^2) = 4\pi \frac{Q}{r^2} \) so \( E_r = \frac{Q}{\kappa r^2} \).

   (b) Find the voltage difference \( V_a - V_b \). Explain your reasoning.

   \[
   V_a - V_b = -\int_b^a \mathbf{E} \cdot d\mathbf{s} = \int_a^b \frac{Q}{\kappa r^2} \, r \, dr
   \]
   \[
   = \int_a^b \frac{Q}{\kappa r^2} \, r \, dr = -\frac{Q}{\kappa} \left[ \frac{1}{r} \right]_a^b = \frac{Q}{\kappa} \left( \frac{1}{a} - \frac{1}{b} \right)
   \]

   (c) Find the capacitance.

   \[
   C = \frac{Q}{\Delta V} = \frac{Q}{V_a - V_b} = \frac{\kappa}{\frac{1}{a} - \frac{1}{b}} = \frac{\kappa (a b)}{b (a - b)}
   \]
3. A parallel plate capacitor with an air gap has electrical energy $1.8 \times 10^{-5}$ J when connected to a 3 V battery. It is now disconnected from the battery. A slab of dielectric constant $\kappa = 4$ and nearly the same thickness as the capacitor is slid into the capacitor.

a. (6 pts) Find the initial and final charges on the plates.

$$U = \frac{1}{2} C(\Delta V)^2 \quad \text{or} \quad U = \frac{1}{2} \frac{Q^2}{C} \quad \text{or} \quad U = \frac{1}{2} Q(\Delta V)$$

Then

$$Q = \frac{2U}{\Delta V} = \frac{2(1.8 \times 10^{-5} \text{J})}{3 \text{V}} = 1.2 \times 10^{-5} \text{C}$$

This is both the initial and final value for the plate charge (on the + side) because charge is conserved and cannot escape the plate while the slab is slid in.

$$Q_i = Q_f = 1.2 \times 10^{-5} \text{C}$$

b. (6 pts) Find the initial and final voltage difference.

$$\Delta V_i = 3 \text{V} \quad \Delta V_f = \frac{\Delta V_i}{\kappa} = \frac{3 \text{V}}{4} = 0.75 \text{V}$$

c. (3 pts) When part way in the capacitor, was the dielectric attracted, repelled, or does it feel no force? Explain or get zero credit.

Attraction, because dielectric gets polarized and is in a non-uniform field, just as with the amber effect (e.g., paper polarized by charged comb)

4. (20 pts) Consider three capacitors. $C_1 = 24 \mu\text{F}$ and $C_2 = 21 \mu\text{F}$ are in parallel, and $C_3 = 15 \mu\text{F}$ is in series with them. $V_a = 3 \text{V}$ and $V_b = -12 \text{V}$. Find the charge and voltage difference for each capacitor. Find $V_a$.

Add charges $Q_1$, $Q_2$, $Q_3$ on Figure.

Add voltages $V_c$ and $V_b$ on Figure.

By charge conservation, $Q_1 + Q_2 = Q_3$.

Also, $C' = C_1 + C_2 = 45 \mu\text{F}$

and $C' \text{ gets } Q_3$.

But $Q_3 = C_3 \Delta V_3 = (15 \mu\text{F})(3 \text{V} - (-12 \text{V}))$

$= (15 \mu\text{F})(15 \text{V}) = 225 \mu\text{C} = Q'$

Then $\Delta V' = \Delta V_1 = \Delta V_2 = \frac{Q'}{C'} = \frac{Q_3}{C'} = \frac{225 \mu\text{C}}{45 \mu\text{F}} = 5 \text{V}$.

Finally $Q_1 = C_1 \Delta V_1 = (24 \mu\text{F})(5 \text{V}) = 120 \mu\text{C}$

and $Q_2 = C_2 \Delta V_2 = (21 \mu\text{F})(5 \text{V}) = 105 \mu\text{C}$

Note that $Q_1 + Q_2 = 225 \mu\text{C} = Q_3$.

Also, $V_a = V_b - 5 \text{V} = -17 \text{V}$
5. (25 pts) For the circuit below, take $E_1 = 4\, \text{V}$, $E_2 = 10\, \text{V}$, $r_1 = 0.02\, \Omega$, $r_2 = 0.03\, \Omega$, $R = 0.05\, \Omega$.

(a) On the figure indicate your definitions of the directions of positive currents and of the positive side of the voltage $\Delta V$ across $R$.

(b) Analyze the circuit using Kirchoff's rules.

Use of current conservation is junction rule

Use of common voltage difference $\Delta V$ is equivalent to loop rule

(c) Solve for the voltage across $R$. $\Delta V = \frac{133.3 A}{103.3 \, \Omega} = 1.29\, \text{V}$

(d) Find the current through $R$ and the currents provided by each of the batteries.

$I = \frac{\Delta V}{R} = \frac{1.29}{0.05} = 25.8\, \text{A}$

$I_1 = \frac{E_1}{r_1} = \frac{4 + 1.29}{0.02} = 264.5\, \text{A}$

$I_2 = \frac{E_2}{r_2} = \frac{10 - 1.29}{0.03} = 290.3\, \text{A}$

6. (20 pts) Find the unknown currents, the unknown resistance, and the unknown emf for the circuit below.

To get $E$, start at bottom left. Voltage changes are:

1) Across $6\, \Omega$, $\Delta V_6 = I_6(6) = 0$.
2) Across $4\, \text{V} \, 2\, \Omega$, $\Delta V = 4\, \text{V} - 4 \times 2 = 4 - 8 = -4$.
3) Across $R$, $\Delta V_R = -I_8 R = -(3)(4) = -12$
4) Across $E_1$, gain $E$. It must compensate the losses $\Delta V_4$ and $12\, \text{V}$, so $E = 16\, \text{V}$
7. (10 pts) Conventional usage doesn’t distinguish between a true voltaic cell and a true battery.  
(a) Distinguish between a true voltaic cell and a true battery.  
A true voltaic cell is one unit consisting of two electrodes and electrolyte within which chemical reaction occurs to produce electricity. A true battery contains more than one voltaic cell.  
(b) Is a car “battery” a voltaic cell or a true battery, and why?  
Car batteries are true batteries. Can’t get 12 V from a single voltaic cell.  
(c) Is a 1.5 V AA “battery” a voltaic cell or a true battery, and why?  
A 1.5V AA "battery" is a true voltaic cell with only one set of electrodes.  
(d) Generically, what kind of energy is stored in “batteries”?

8. (15 pts) A 20 cm long rod with radius 2 mm carries 4 A when a voltage difference of 0.5 V (high voltage to left) is placed across its ends.  
(a) Find the resistivity.  
\[ R = \frac{\rho}{A} \quad \text{and} \quad R = \frac{\Delta V}{I} \quad \text{so} \quad \frac{\rho}{A} = \frac{\Delta V}{I} \]

\[ \rho = \frac{A \Delta V}{I} = \frac{(\pi r^2) \Delta V}{I} = 7.85 \times 10^{-6} \Omega \cdot m \]

(b) Find the electric field \( \vec{E} \) within the rod, including direction.  
\[ |\vec{E}| = \frac{\Delta V}{l} = \frac{0.5 V}{0.02 m} = 25 \frac{V}{m} \quad \text{point from high} \ V \ \text{to low} \ V \text{or} \rightarrow \]

\[ \vec{E} = 25 \frac{V}{m} \uparrow \]

(c) Estimate the drift velocity \( \vec{v}_d \) of the charge-carriers, taken to be of electrons of density \( n = 4.5 \times 10^{28} \text{m}^{-3} \). Give its direction.  
\[ \vec{j} = n q \vec{v}_d \quad \text{is along} \ \vec{E}, \ \text{so it points} \rightarrow \]

\[ \vec{j} = n e \vec{v}_d \quad \text{and} \quad |\vec{j}| = \frac{I}{A} \quad \text{so} \quad ne \vec{v}_d = \frac{I}{A} \]

Hence \( \vec{v}_d = \frac{I}{neA} = \frac{4.42 \times 10^{-5} \text{m/s}}{I} \). Since \( q = -e \), \( \vec{v}_d = \frac{\vec{j}}{-ne} \) is to left.

9. (15 pts) A voltaic cell has fixed emf \( \mathcal{E} \) and internal resistance \( r \). It is part of a series circuit with a resistor \( R \).  
(a) For what value of \( R \) will the power \( P \) to \( R \) be a maximum?  
\[ R = r \quad \text{(Impedance matching)} \]

\[ P_{\text{max}} = I^2 R = \left( \frac{\mathcal{E}}{2r} \right)^2 R = \left( \frac{\mathcal{E}}{2r} \right)^2 r = \left( \frac{\mathcal{E}}{4r} \right) R = \frac{\mathcal{E}^2}{4r} \]

(b) Derive that maximum power \( P_{\text{max}} \).

(c) For general \( R \), find the power \( P \) to \( R \).

\[ \mathcal{P} = I^2 R = \frac{\mathcal{E}^2 R}{r + R} \]

(d) Plot \( \mathcal{P} \) vs \( R \).
10. The capacitor is uncharged initially. The switch is then closed at \( t = 0 \). Let \( \mathcal{E} = 6 \text{ V}, \quad r = 1 \text{ \Omega}, \quad R_1 = 6 \text{ \Omega}, \quad R_2 = 3 \text{ \Omega}, \quad C_1 = 6 \mu \text{ F} \).

a. (8 pts) Find \( I_1, Q_1, I_1 \), and \( I_2 \) just after the switch is closed. Explain.

\[ R_{\text{eff}} = r + \left( \frac{1}{R_1} + \frac{1}{R_2} \right)^{-1} = 1 + \left( \frac{1}{6} + \frac{1}{3} \right)^{-1} = 1 + \frac{3}{5} = 1 + \frac{3}{2} = \frac{5}{2} \text{ \Omega} \]

\[ I = \frac{\mathcal{E}}{R_{\text{eff}}} = \frac{6}{\frac{5}{2}} = \frac{12}{5} \text{ A} \]

\[ I_1 = \frac{\mathcal{E}}{R_1} = \frac{6}{6} = 1 \text{ A} \]

\[ I_2 = I - I_1 = \frac{12}{5} - 1 = \frac{7}{5} \text{ A} \]

b. (8 pts) Find \( I, Q_1, I_1 \), and \( I_2 \) long after the switch is closed. Explain.

After a long time \( I_1 \rightarrow 0 \), so \( I = I_r = I_2 = \frac{\mathcal{E}}{r + R_2} = \frac{6}{1 + 3/2} = \frac{12}{5} = 2.4 \text{ A} \)

\[ \Delta V_1 = \Delta V_2 = \Delta V_{C_1}, \quad \text{so} \quad I_2 R_2 = \frac{0}{C_1} \quad \text{or} \quad Q_1 = I_2 R_2 C_1 = (1.5)(3)(6 \mu \text{ F}) = 27 \mu \text{ C} \]

c. (4 pts) Sketch \( I_2 \) as a function of time.

11. A voltaic cell has internal resistance \( r = 0.15 \text{ \Omega} \) and open circuit voltages across the left and right electrodes of 0.4 V and 1.4 V, for a net emf of \( \mathcal{E} = 1.8 \text{ V} \). It is in series with a resistor \( R = 0.45 \text{ \Omega} \). Let \( V_i = 0.3 \text{ V} \). The connecting wires have zero resistance.

a. (15 pts) Find the current, the voltage drops across the resistances, and sketch the voltage around the circuit. (Hint: start from point 1.)

\[ I = \frac{\mathcal{E}}{R + r} = \frac{1.8}{0.45} = 4 \text{ A} \]

\[ I_r = 3(0.15) = 0.45 \text{ V} \]

\[ I R = 3(0.45) = 1.35 \text{ V} \]

b. (5 pts) If the voltaic cell discharges in 40 minutes, find its initial "charge" and its initial energy.

\[ Q = I t = (3)(0.675) = 2 \text{ A} \cdot \text{hr} = 7200 \text{ C} \]

\[ U = \frac{Q}{E} \cdot E = \frac{7200}{3} \cdot 3 = 21600 \text{ J} \]

Note: If you took \( |\mathcal{E}_{\text{left}}| = 0.4 \text{ V} \) and \( |\mathcal{E}_{\text{right}}| = 1.4 \text{ V} \), then the sizes of the plates should switch and the current would go to 0. Then the voltage across the plates would look like this. Full credit if done like this.