(1) [25 points] Two wide, flat conducting plates are arranged as shown, with inner surfaces (area \( A \)) parallel to the \( x-z \) plane, at positions \( y = \pm a \). Assume that the plates are wide enough to allow edge effects to be neglected, so the fields have \( y \)-dependence only. The region between the plates is filled with a non-conducting material having volume charge \( \rho = \rho_0 \sin \frac{\pi y}{a} \), with \( \rho_0 \) a constant.

(a) Find a general relationship for the electric field between the plates.

(b) If the boundary conditions are set so that the electric field is given by \((+\frac{\rho_0 a}{\varepsilon_0})\) at \( y = \pm a \), just inside the surface of each plate, find the potential difference between the two plates.

(c) Find the total charge (with sign) on each plate.

(d) If the plate on the right is moved to the right, opening a gap between it and the charged region, while maintaining the total charges on the plates unchanged, describe the resulting electric field between the plates.

\[
\begin{align*}
\n\n(a) \quad \vec{E} &= \frac{\rho}{\varepsilon_0} \\
\quad \frac{dE_y}{dy} &= \text{constant} \\
\text{so} \quad E_y &= -\frac{\rho_0}{\varepsilon_0} \cos \frac{\pi y}{a} + C \\
(b) \quad \text{at} \quad y = \pm a \quad \cos \frac{\pi y}{a} = -1 \quad \text{so} \\
\quad E_y &= \frac{\rho_0}{\varepsilon_0} + C \\
\quad \text{Thus} \quad E &= \frac{\rho_0}{\varepsilon_0} \left( 1 - \frac{1}{\pi} + \frac{1}{\pi} \cos \frac{\pi y}{a} \right) \\
\quad \Delta V &= -\int_{-a}^{a} E_y \, dy = \frac{-\rho_0}{\varepsilon_0} \int_{-a}^{a} (1 - \frac{1}{\pi}) dy + \frac{\rho_0}{\varepsilon_0} \int_{-a}^{a} \cos \frac{\pi y}{a} \, dy \\
\quad \text{limits} \\
\quad \Delta V &= -2\rho_0 a^2 \left( \frac{\pi - 1}{\pi} \right) \quad \text{[plate on right is more negative]}
\end{align*}
\]

(c) By Gauss's law, \( \sigma = \varepsilon_0 E \) at surface of conductor;

\( \quad Q = \varepsilon_0 EA = \varepsilon_0 A \rho_0 a / \varepsilon_0 = \rho_0 a A \)

\( \rightarrow \) left face \(+\rho_0 a A\), right face \(-\rho_0 a A\).

(d) \( \vec{E} \) in gap will be uniform \& equal to \( \hat{\jmath} \rho_0 a / \varepsilon_0 \), same as boundary condition \( E \)-field above. Also for \(-a < y < a\), \( E \) is unchanged since charge have not rearranged.

Can show by Gauss's law argument. \( \boxed{ \int_{\text{gauge}} \vec{E} \cdot \hat{A} \, dA } \) use such a gaussian}
(2) [25 points] An electric field is given (in cylindrical coordinates \(z, s, \phi\)) by
\[
\vec{E} = z[A \cos \phi - \hat{\phi} (A \sin \phi) / s],
\]
in some region of space, where \(A\) is a constant.
(a) Determine whether this field could be found in an electrostatic situation.
(b) If this field is allowed in electrostatics, determine the charge density in this region.
(c) Similar to the above, suppose in spherical coordinates, \(\vec{E} = \hat{\theta} [B \sin \theta] - \hat{\phi} [B \cos \theta]\), in some region with \(B\) a constant. Determine whether this field can exist in an electrostatic situation.
(d) If the field from (c) is allowed, determine the charge density in the region where it occurs.
(e) Choose one of these fields, which is allowed in the static situation, and determine the potential difference in going from the point \((x = a, y = 0, z = 0)\) to \((x = 0, y = 2a, z = 0)\). These are cartesian coordinates; you will need to translate to the appropriate curvilinear coordinates.

(a) To check, \(\nabla \times \vec{E} = 0\)? Using attached vector formulas,
\[
\nabla \times \vec{E} = \hat{\phi} \left[ -\frac{1}{s} A \sin \phi + \frac{A \sin \phi}{s} \right] + \hat{\theta} \left[ 0 - 0 \right] + \hat{r} \left[ 0 - 0 \right] = 0
\]
so \(\nabla \times \vec{E} = 0\) and \(\vec{E}\) qualifies as allowed field.

(b) Gauss' law \(\Rightarrow p = \varepsilon_0 \nabla \cdot \vec{E} = \varepsilon_0 \left( -\frac{1}{s^2} 2A \cos \phi \right) = -\frac{\varepsilon_0 A z \cos \phi}{s^2}
\]

(c) \(\nabla \times \vec{E} = \hat{\phi} \left[ 0 - 0 \right] + \hat{\theta} \left[ 0 - 0 \right] + \hat{r} \left[ B \sin \theta - B \sin \theta \right] = 0\)
more, in Cartesian \(\vec{E}\) reduces to \(\vec{E} = -B \hat{\phi}\)
so not surprising that this fields is OK.

(d) \(p = \varepsilon_0 \nabla \cdot \vec{E} = \varepsilon_0 \frac{1}{r^2} (2 r B \cos \theta) + \frac{\varepsilon_0}{\rho \sin \theta} B 2 \sin \theta \rho \omega \omega = 0\)
so \(p = 0\) (same comment as above, uniform field has \(p = 0\).)

(e) Either field qualifies, and \(\Delta V = \int \vec{E} \cdot d\ell\)
e.g. field (a), we go from \(s = a, \phi = 0\) to \(s = 2a, \phi = \frac{\pi}{2}\),
\(\Delta \phi = \frac{\pi}{2}, s = a\) followed
by radial traverse, \(s = a \Rightarrow 2a\)
arc: \(-\int \vec{E} \cdot d\ell = - (E_\phi)(s \Delta \phi) = + (A \cos \phi) (a \frac{\pi}{2}) = 0 \text{ since } z = 0\)

\(\Delta V = 0\) (same answer for other field).
(3) [25 points] Two concentric spherical shells are arranged as follows: the inner shell (radius \( a \)) is conducting, and is held at a potential \( V = 0 \), while the outer shell (radius \( b \)) has a potential given in spherical coordinates by \( (V_0 \cos \theta) \), with \( V_0 \) a constant. Between the shells is vacuum, with no charges.

(a) Write the general solutions of the Laplace equation which would apply between \( a \) and \( b \).

(b) State the boundary conditions, and solve to find the potential, \( V(\vec{r}) \), in this region.

(c) Find the surface charge density vs. \( \theta \) at \( r = a \), on the surface of the inner sphere.

\[
\begin{align*}
(a) \quad V(\vec{r}) = \sum_{l=0}^{\infty} \left( \frac{A_l}{r^{l+1}} + \frac{B_l}{r^l} \right) P_l(\cos \theta) \quad \text{is general case (no } \phi \text{-dependence), with } A_l^\text{is} \quad \text{a set of constants.}
\end{align*}
\]

\[
(b) \quad \text{at } r = a, \quad \sum_{l=0}^{\infty} \left( \frac{A_l}{a^{l+1}} + \frac{B_l}{a^l} \right) P_l(\cos \theta) = 0.
\]

but by orthogonality of \( P_l \), coefficients must vanish separately: \( \frac{A_l}{a^{l+1}} + \frac{B_l}{a^l} = 0 \rightarrow B_l = -\frac{A_l}{a^{l+1}} \) each \( l \).

\[
(b) \quad \text{at } r = b, \quad V = \sum_{l=0}^{\infty} \left( \frac{A_l}{b^{l+1}} - \frac{B_l}{a^{l+1}} \right) P_l(\cos \theta) = V_0 \cos \theta\text{,}
\]

\[
\Rightarrow \quad \sum_{l=0}^{\infty} \frac{A_l}{b^{l+1}} \left( 1 - \frac{b^2}{a^2} \right)^l = V_0 P_1(\cos \theta)
\]

orthogonality \( \rightarrow \) \( P_l \) terms satisfy equality individually; but since \( 1 - \left( \frac{b}{a} \right)^{2l+1} \neq 0 \), \( A_l = 0 \) for \( l \neq 1 \).

\[
\text{case of } l = 1: \quad \frac{A_1}{b^2} \left( 1 - \frac{b^2}{a^2} \right) = V_0 \Rightarrow A_1 = -\frac{V_0 b^2}{(a^2 - b^2)}
\]

\[
A_1 = -\frac{V_0 b^2}{(a^2 - b^2)}, \quad \beta_1 = +\frac{V_0 b^2}{(b^2 - a^2)}
\]

\[
\quad V(\vec{r}) = \frac{V_0 b^2}{b^2 - a^2} \left( r - \frac{a^2}{r^3} \right) \cos \theta.
\]

\[
(c) \quad E_r = -\frac{\partial V}{\partial r} = -\frac{V_0 b^2}{b^2 - a^2} \left( 1 + \frac{2a^2}{r^3} \right) \cos \theta \quad \Rightarrow \quad -3\frac{V_0 b^2 \cos \theta}{b^2 - a^2} \quad (r = a)
\]

\[
\Rightarrow \quad \sigma = \varepsilon_0 E_r = -3\varepsilon_0 V_0 \frac{b^2 \cos \theta}{b^2 - a^2}.
\]
(4) [25 points] A point charge \( +Q \) is located close to a grounded “corner reflector”, consisting of three perpendicular metal plates. The charge is equidistant from the plates, with \( d \) the distance to each plate. The sketch is a cutaway view, but consider the planes to extend to infinity. [Note, the distance of \( Q \) to the corner would be \( d\sqrt{3} \].]

(a) Describe the charge distribution used to solve this problem.
(b) Find the force on the charge \( +Q \), with direction.
(c) Find the total charge on each of the three surfaces. This does not require a calculation, but you should explain your reasoning sufficient to show how you got the result.
(d) Find the net force on each plate, with direction. (You can state the magnitude relative to the part (b) answer. Hint—if you are doing a long calculation for this one you are off-track.)

(a) Need \( +Q \) and \( -Q \) image charges, so that the 8 total charges are on corners of a cube centered at the origin; all charges \( \pm Q \).

(b) Force due to 3 adjacent \( (-Q) \) charges: \( F = \frac{Q^2}{4\pi \varepsilon_0 (2d)^2} \) each

\[ F = \frac{Q^2}{4\pi \varepsilon_0 d^2} (\sqrt{3}/4) \text{ points toward corner } \]

3 further \( (+Q) \) charges: each \( F = \frac{Q^2}{4\pi \varepsilon_0} \cdot \frac{1}{8d^2} \)

each of 3 has equal \( (x, y), (x, z), \) or \( (y, z) \) components,
equal to \( \frac{1}{\sqrt{2}} \times \frac{Q^2}{4\pi \varepsilon_0} \frac{1}{8d^2} \)

result is \( F_x = F_y = F_z = \frac{2}{\sqrt{2}} \frac{Q^2}{8d^2} (4\pi \varepsilon_0) \Rightarrow |F| = \frac{\sqrt{3}Q^2}{4\pi \varepsilon_0 - 4d^2 \sqrt{2}} \)

(points away from corner)

last charge: far corner, \( |F| = \frac{Q^2}{4\pi \varepsilon_0 (2d)^2} \)
\[ Net = \frac{Q^2}{4\pi \varepsilon_0} \left( \frac{\sqrt{3}}{4} - \frac{1}{4\sqrt{2}} + \frac{1}{12} \right) \text{ points toward corner.} \]
cont’d: Note that the net polarization charge induced is \((-Q)\), and the \((+Q)\) charge will be attracted to the surface polarized charges; by symmetry this will point towards the corner; can thus get the direction from a general argument.

(c) \((\text{Charge} + \text{image charges})\) makes up a system with no net charge, so field lines will all start \& end inside. That means all the \((+Q)\) field lines actually end on the surfaces, so from Gauss’ law we know that \((-\frac{Q}{3}) = \text{net surface charge}.\)

Thus by symmetry \((-\frac{Q}{3})\) resides on each surface.

(d) Call \(\mathbf{F}_a\) the force on charge \((+Q)\) from (c).

- \(\mathbf{F} = -\mathbf{F}_a\) must be total force on 3 surfaces.
  (3rd law)
- But each surface force is perpendicular since \(E \perp\) metal surface.
- So \(\mathbf{F} = -\mathbf{F}_a\) is resultant of 3 \(\perp\) forces.
  some geometry \(\Rightarrow\) each surface \(1F_1 = \frac{1}{\sqrt{3}} 1F_a\), with direction \(\perp\) surface as above.