Wave Equations:

(1) Strings, EM waves, pressure, etc:

- Wave equation: \[ \frac{\partial^2 \psi}{\partial x^2} = \left(\frac{1}{v^2}\right) \frac{\partial^2 \psi}{\partial t^2} \]
  - linear equation; superposition works
  - hence interference phenomena
  - \( \psi \): represents displacement, \( E & B \) field, etc.

- Solutions of form: \( \psi = A \sin(kx \mp \omega t) \)

  \( k = \frac{2\pi}{\lambda} \) is wave number [wave vector in 3d]
  \( \omega = \frac{2\pi}{T} = 2\pi f \) angular frequency (rad/s)
  \( v = \frac{\omega}{k} \) wave velocity.

  \( u_{ph} = \frac{\omega}{k} ; \quad u_{gr} = \frac{d\omega}{dk} \) group and phase velocities

(2) Schrödinger wave equation, for matter waves.

- Wave equation: \[ i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V\psi \]
  - linear equation; superposition also.
  - \( \psi \): connected to probability;
    \[ |\psi|^2 = \psi^* \psi \] is probability density.
Waves

\[ \psi = \psi(x, t) \quad \text{wave function} \]

\[ v = \sqrt{\frac{F}{\mu}} \]

LIGHT

\[ \psi = E^*(x, t) \quad \text{[and } B(x, t)\text{]} \]

- interference = superposition
- waves don't interact

MATTER WAVES

\[ |\psi|^2 = \sqrt{\text{probability}} \]

\[ v \text{ not fixed} \]
Wave on a string

Straight, no net force.

Curved; force towards inside of curve.

mass \( (\mu \, dx) \).

\[
F_y = F \sin \theta \approx F \tan \theta = F \frac{dy}{dx}
\]

Net:

\[
F_y = F \frac{dy(x + dx)}{dx} - F \frac{dy(x)}{dx}
\]

\[
= F \frac{d^2 y}{dx^2} \, dx
\]
Wave on a string

\[ F_y = F \frac{d^2 y}{dx^2} \, dx \]

mass \((\mu \, dx)\).

\[ F_y = m a_y \quad \Rightarrow \quad F \frac{d^2 y}{dx^2} \, dx = (\mu \, dx) \frac{d^2 y}{dt^2} \]

This is wave equation
(1) Traveling waves:

\[ \psi = A \cos(kx \mp \omega t) = A \cos \left[ \frac{2\pi x}{\lambda} \mp \frac{2\pi \omega}{T} \right] \]

- no nodes, wave sweeps uniformly through space.
- \( v_x = \pm \omega / k \)
- 3D version, \( \psi = A \cos(\vec{k} \cdot \vec{r} \mp \omega t) \)
  \( \vec{k} \) is wave vector

(2) Standing waves:

\[ \psi = A \left[ \cos(kx + \omega t) + \cos(kx - \omega t) \right] \]

\[ = 2A \cos(kx) \cos(\omega t) \]

- Superposition with same \( A \), opposite velocities.

(3) Beats:

\[ \psi = A \left[ \cos(k_1 x - \omega_1 t) + \cos(k_2 x - \omega_2 t) \right] \]

\[ = 2A \cos \left[ \frac{k_1 + k_2}{2} x - \frac{\omega_1 + \omega_2}{2} t \right] \cos \left[ \frac{k_1 - k_2}{2} x - \frac{\omega_1 - \omega_2}{2} t \right] \]

- Superposition with different frequencies
  (closely spaced).
- Envelope function moves with group velocity
Standing wave = periodic wave traveling both directions

(a) String is one-half wavelength long.
(b) String is one wavelength long.
(c) String is one and a half wavelengths long.
(d) String is two wavelengths long.
(e) The shape of the string in (b) at two different instants.

N = nodal points at which the string never moves.
A = antinodal points at which the amplitude of string motion is greatest.

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Wave Interference and Beats
Wave-packet approximation

gaussian approximation

wave interference, fine frequencies
Wave-packet approximation

gaussian approximation

wave interference, mix frequencies
Consider a square wave \( f(x) \) of length \( 2L \). Over the range \([0, 2L]\), this can be written as

\[
f(x) = 2 \left[ H(x/L) - H(x/L - 1) \right].
\]

(1)

where \( H(x) \) is the Heaviside step function. Since \( f(x) = f(2L - x) \), the function is odd, so \( a_n = 0 \), and

\[
b_n = \frac{1}{L} \int_0^{2L} f(x) \sin \left( \frac{n \pi x}{L} \right) \, dx
\]

reduces to

\[
b_n = \frac{2}{L} \int_0^{L} f(x) \sin \left( \frac{n \pi x}{L} \right) \, dx
\]

(3)

\[
= \frac{4}{n \pi} \sin^2 \left( \frac{1}{2} n \pi \right)
\]

(4)

\[
= \frac{2}{n \pi} \left[ 1 - (-1)^n \right]
\]

(5)

\[
= \frac{4}{n \pi} \left( \text{if } n \text{ even} \right)
\]

(6)

\[
= \frac{1}{n \pi} \left( \text{if } n \text{ odd} \right)
\]

The Fourier series is therefore

\[
f(x) = \frac{4}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n} \sin \left( \frac{n \pi x}{L} \right).
\]

(7)

SEE ALSO: Fourier Series, Fourier Series—Sawtooth Wave, Fourier Series—Triangle Wave, Gibbs Phenomenon, Square Wave

CITE THIS AS:

Fourier Transform--Gaussian

The Fourier transform of a Gaussian function $f(x) \equiv e^{-ax^2}$ is given by

$$\mathcal{F}[e^{-ax^2}](k) = \int_{-\infty}^{\infty} e^{-ax^2} e^{-2\pi ikx} dx$$

$$= \int_{-\infty}^{\infty} e^{-ax^2} [\cos(2 \pi kx) - i \sin(2 \pi kx)] dx$$

$$= \int_{-\infty}^{\infty} e^{-ax^2} \cos(2 \pi kx) dx - i \int_{-\infty}^{\infty} e^{-ax^2} \sin(2 \pi kx) dx. \quad (3)$$

The second integrand is odd, so integration over a symmetrical range gives 0. The value of the first integral is given by Abramowitz and Stegun (1972, p. 302, equation 7.4.6), so

$$\mathcal{F}[e^{-ax^2}](k) = \sqrt{\frac{\pi}{a}} e^{-\pi^2 k^2/a}. \quad (4)$$

so a Gaussian transforms to another Gaussian.

SEE ALSO: Gaussian Function, Fourier Transform

REFERENCES:


CITE THIS AS:

Fourier Transform of Gaussian Function

Continuous Fourier transform of Gaussian function is another Gaussian (see, e.g. mathworld.com):

$$
\int_{-\infty}^{\infty} e^{-\frac{k^2}{2\Delta k^2}} \cos kx \, dx = \sqrt{2\pi\Delta k} e^{-\frac{2\pi^2(\Delta k^2)x^2}{2}}
$$

- Note: spectral width is inverse of spatial width -- wider distribution of wavevectors $k$ gives narrower peak in $x$.
- This is related to the uncertainty principle: $\Delta k\Delta x \sim 1$ can be converted to $\Delta x\Delta p \sim \hbar$ using DeBroglie wavelength relation, $\lambda = \hbar/p$ and $k = \frac{2\pi}{\lambda}$.

By a substitution of variables:

$$
\int_{-\infty}^{\infty} e^{-\frac{(k-k_0)^2}{2\Delta k^2}} \cos kx \, dx = \sqrt{2\pi\Delta k} e^{-\frac{2\pi^2(\Delta k^2)x^2}{2}} \cos(k_0x + \phi)
$$

(The phase angle $\phi$ is readily determined.)
Thus a Gaussian centered at $k_0$ is a wavepacket with a Gaussian envelope.
[Mathematically this is also an example of the convolution theorem.]