USEFUL FORMULAE IN DIFFERENTIAL GEOMETRY

Differential forms:
\[ \alpha = \frac{1}{p!} \alpha_{\mu_1...\mu_p} dx^{\mu_1} \land \cdots \land dx^{\mu_p}; \quad \alpha \in \land^p. \] (1)
\[ \alpha \land \beta = (-)^{pq} \beta \land \alpha; \quad \alpha \in \land^p, \quad \beta \in \land^q. \] (2)

Exterior derivative, \( d \):
\[ d\alpha \equiv \frac{1}{p!} \partial_v [\alpha_{\mu_1...\mu_p]} dx^v \land dx^{\mu_1} \land \cdots \land dx^{\mu_p}. \] (3)
d maps \( p \)-forms to \( (p+1) \)-forms:
\[ d : \land^p \rightarrow \land^{p+1}; \quad d^2 = 0. \] (4)

Defining the components of \( d\alpha \), \( (d\alpha)_{\mu_1...\mu_{p+1}} \), by
\[ d\alpha \equiv \frac{1}{(p+1)!} (d\alpha)_{\mu_1...\mu_{p+1}} dx^{\mu_1} \land \cdots \land dx^{\mu_{p+1}}, \] (5)
we have
\[ (d\alpha)_{\mu_1...\mu_{p+1}} = (p+1) \partial_{[\mu_1} \alpha_{\mu_2...\mu_{p+1}]}, \] (6)
where
\[ T_{[\mu_1...\mu_q]} \equiv \frac{1}{q!} \left( T_{\mu_1...\mu_q} + \text{even perms} - \text{odd perms} \right). \] (7)

Leibnitz rule:
\[ d(\alpha \land \beta) = d\alpha \land \beta + (-)^p \alpha \land d\beta, \quad \alpha \in \land^p, \quad \beta \in \land^q. \] (8)

Stokes’ Theorem:
\[ \int_M d\omega = \int_{\partial M} \omega, \] (9)
where \( M \) is an \( n \)-manifold and \( \omega \in \land^{n-1} \).

Epsilon tensors and densities:
\[ \varepsilon_{\mu_1...\mu_n} \equiv (+1, -1, 0) \] (10)
if \( \mu_1...\mu_n \) is an (even, odd, no) permutation of a lexical ordering of indices \( (1...n) \). It is a tensor density of weight +1. We may also define the quantity \( \varepsilon^{\mu_1...\mu_n} \), with components given numerically by
\[ \varepsilon^{\mu_1...\mu_n} \equiv (-1)^t \varepsilon_{\mu_1...\mu_n}, \]
where \( t \) is the number of timelike coordinates. NOTE: This is the only quantity where we do not raise and lower indices using the metric tensor. \( \varepsilon^{\mu_1 \cdots \mu_n} \) is a tensor density of weight \(-1\). We define epsilon tensors:

\[
\varepsilon_{\mu_1 \cdots \mu_n} = \sqrt{|g|} \varepsilon_{\mu_1 \cdots \mu_n}, \quad \varepsilon^{\mu_1 \cdots \mu_n} = \frac{1}{\sqrt{|g|}} \varepsilon^{\mu_1 \cdots \mu_n},
\]

(11)

where \( g \equiv \det(g_{\mu \nu}) \) is the determinant of the metric tensor \( g_{\mu \nu} \). Note that the tensor \( \varepsilon^{\mu_1 \cdots \mu_n} \) is obtained from \( \varepsilon_{\mu_1 \cdots \mu_n} \) by raising the indices using inverse metrics.

Epsilon-tensor identities:

\[
\varepsilon_{\mu_1 \cdots \mu_n} \varepsilon^{\nu_1 \cdots \nu_n} = (-1)^t n! \delta^{\nu_1 \cdots \nu_n}_{\mu_1 \cdots \mu_n}.
\]

(12a)

From this, contractions of indices lead to the special cases

\[
\varepsilon_{\mu_1 \cdots \mu_r \mu_{r+1} \cdots \mu_n} \varepsilon^{\nu_1 \cdots \nu_r \nu_{r+1} \cdots \nu_n} = (-1)^t r!(n - r)! \delta^{\nu_1 \cdots \nu_n}_{\mu_1 \cdots \mu_n},
\]

(12b)

where again \( t \) denotes the number of timelike coordinates. The multi-index delta-functions have unit strength, and are defined by

\[
\delta^{\nu_1 \cdots \nu_p}_{\mu_1 \cdots \mu_p} \equiv \delta^{[\nu_1 \cdots \nu_p]}_{[\mu_1 \cdots \mu_p]}.
\]

(13)

(Note that only one set of square brackets is actually needed here; but with our “unit-strength” normalisation convention (7), the second antisymmetrisation is harmless.) It is worth pointing out that a common occurrence of the multi-index delta-function is in an expression like \( B_{\nu_1} A_{\nu_2 \cdots \nu_p} \delta^{\nu_1 \cdots \nu_p}_{\mu_1 \cdots \mu_p} \), where \( A_{\nu_2 \cdots \nu_p} \) is totally antisymmetric in its \((p - 1)\) indices. It is easy to see that this can be written out as the \( p \) terms

\[
B_{\nu_1} A_{\nu_2 \cdots \nu_p} \delta^{\nu_1 \cdots \nu_p}_{\mu_1 \cdots \mu_p} = \frac{1}{p} \left( B_{\mu_1} A_{\nu_2 \cdots \nu_p} + B_{\mu_2} A_{\nu_3 \cdots \mu_p \mu_1} + B_{\mu_3} A_{\mu_4 \cdots \mu_p \mu_1 \mu_2} + \cdots + B_{\mu_p} A_{\mu_1 \cdots \mu_{p-1}} \right)
\]

if \( p \) is odd. If instead \( p \) is even, the signs alternate and

\[
B_{\nu_1} A_{\nu_2 \cdots \nu_p} \delta^{\nu_1 \cdots \nu_p}_{\mu_1 \cdots \mu_p} = \frac{1}{p} \left( B_{\mu_1} A_{\mu_2 \cdots \mu_p} - B_{\mu_2} A_{\mu_3 \cdots \mu_p \mu_1} + B_{\mu_3} A_{\mu_4 \cdots \mu_p \mu_1 \mu_2} - \cdots - B_{\mu_p} A_{\mu_1 \cdots \mu_{p-1}} \right).
\]

Hodge \(*\) operator:

\[
*(dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}) \equiv \frac{1}{(n - p)!} \varepsilon_{\nu_1 \cdots \nu_{n-p}}^{\mu_1 \cdots \mu_p} dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_{n-p}}.
\]

(14)

The Hodge *, or dual, is thus a map from \( p \)-forms to \((n - p)\)-forms:

\[
*: \wedge^p \to \wedge^{n-p}.
\]

(15)

Note in particular that taking \( p = 0 \) in (14) gives

\[
*1 = \epsilon = \frac{1}{n!} \varepsilon_{\mu_1 \cdots \mu_n} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n} = \sqrt{|g|} dx^1 \wedge \cdots \wedge dx^n.
\]

(16)
This is the general-coordinate-invariant volume element \( \sqrt{|g|} \, d^n x \) of Riemannian geometry. It should be emphasised that conversely, we have

\[
dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_n} = (-1)^t \varepsilon^{\mu_1 \mu_2 \cdots \mu_n} \, d^n x = (-1)^t \epsilon^{\mu_1 \mu_2 \cdots \mu_n} \sqrt{|g|} \, d^n x.
\]

This extra \((-1)^t\) factor is tiresome, but unavoidable if we want our definitions to be such that \(*1\) is always the positive volume element.

From these definitions it follows that

\[
* \alpha \wedge \beta = \frac{1}{p!} |\alpha \cdot \beta| \epsilon,
\]

where \(\alpha\) and \(\beta\) are \(p\)-forms and

\[
|\alpha \cdot \beta| \equiv \alpha_{\mu_1 \cdots \mu_p} \beta^{\mu_1 \cdots \mu_p}.
\]

Applying \(\ast\) twice, we get

\[
\ast \ast \omega = (-)^{p(n-p)+t} \omega, \quad \omega \in \wedge^p.
\]

In even dimensions, \(n = 2m\), \(m\)-forms can be eigenstates of \(\ast\), and hence can be self-dual or anti-self-dual, in cases where \(\ast \ast = +1\). From (19), we see that this occurs when \(m\) is even if \(t\) is even, and when \(m\) is odd if \(t\) is odd. In particular, we can have real self-duality and anti-self-duality in \(n = 4k\) Euclidean-signature dimensions, and in \(n = 4k + 2\) Lorentzian-signature dimensions.

**Adjoint operator, \(\delta\):**

First define the inner product

\[
(\alpha, \beta) \equiv \int_M \ast \alpha \wedge \beta = \frac{1}{p!} \int_M |\alpha \cdot \beta| \epsilon = (\beta, \alpha),
\]

where \(\alpha\) and \(\beta\) are \(p\)-forms. Then \(\delta\), the adjoint of the exterior derivative \(d\), is defined by

\[
(\alpha, d\beta) \equiv (\delta \alpha, \beta),
\]

where \(\alpha\) is an arbitrary \(p\)-form and \(\beta\) is an arbitrary \((p-1)\)-form. Hence

\[
\delta \alpha = (-)^{np+t} \ast d \ast \alpha, \quad \alpha \in \wedge^p.
\]

(We assume that the boundary term arising from the integration by parts gives zero, either because \(M\) has no boundary, or because appropriate fall-off conditions are imposed on the fields.)

\(\delta\) is a map from \(p\)-forms to \((p-1)\)-forms:

\[
\delta : \wedge^p \rightarrow \wedge^{p-1}; \quad \delta^2 = 0.
\]
Note that in Euclidean signature spaces, $\delta$ on $p$-forms is given by
\begin{align*}
\delta \alpha &= \ast d \ast \alpha, \quad \text{if at least one of } n \text{ and } p \text{ even,} \\
\delta \alpha &= -\ast d \ast \alpha, \quad \text{if } n \text{ and } p \text{ both odd.} \tag{24}
\end{align*}

The signs are reversed in Lorentzian spacetimes.

In terms of components, the above definitions imply that for all spacetime signatures, we have
\begin{align*}
\delta \alpha &= \frac{1}{(p-1)!} (\nabla_\nu \alpha^{\nu \mu_1 \ldots \mu_{p-1}}) \, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{p-1}}, \tag{25}
\end{align*}
where
\begin{align*}
\nabla_\nu \alpha^{\nu \mu_1 \ldots \mu_{p-1}} &\equiv \frac{1}{\sqrt{g}} \partial_\nu \left( \sqrt{g} \alpha^{\nu \mu_1 \ldots \mu_{p-1}} \right) \tag{26}
\end{align*}
is the covariant divergence of $\alpha$. Defining the components of $\delta \alpha$, $(\delta \alpha)_{\mu_1 \ldots \mu_{p-1}}$, by
\begin{align*}
\delta \alpha &\equiv \frac{1}{(p-1)!} (\delta \alpha)_{\mu_1 \ldots \mu_{p-1}} \, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{p-1}}, \tag{27}
\end{align*}
we have
\begin{align*}
(\delta \alpha)_{\mu_1 \ldots \mu_{p-1}} &= -\nabla_\nu \alpha^{\nu \mu_1 \ldots \mu_{p-1}}. \tag{28}
\end{align*}

**Hodge-de Rham operator:**
\begin{align*}
\Delta &\equiv d \delta + \delta d = (d + \delta)^2. \tag{29}
\end{align*}
\(\Delta\) maps $p$-forms to $p$-forms:
\begin{align*}
\Delta : \wedge^p &\to \wedge^p. \tag{30}
\end{align*}
On 0-, 1-, and 2-forms, we have
\begin{align*}
\text{0-forms:} &\quad \Delta \phi = -\nabla_\lambda \nabla^\lambda \phi, \\
\text{1-forms:} &\quad \Delta \omega_\mu = -\nabla_\lambda \nabla^\lambda \omega_\mu + R_\mu^\nu \omega_\nu, \tag{31} \\
\text{2-forms:} &\quad \Delta \omega_{\mu\nu} = -\nabla_\lambda \nabla^\lambda \omega_{\mu\nu} - 2R_{\mu\rho\nu\sigma} \omega^{\rho\sigma} + R_\mu^\sigma \omega_\sigma_\nu - R_\nu^\sigma \omega_\sigma_\mu,
\end{align*}
where $R_{\mu\nu\rho\sigma}$ is the Riemann tensor and
\begin{align*}
R_{\mu\nu} &\equiv R^{\rho}_{\mu\rho\nu} \tag{32}
\end{align*}
is the Ricci tensor.

**Hodge’s theorem:**
We can uniquely decompose an arbitrary $p$ form $\omega$ as
\begin{align*}
\omega = d \alpha + \delta \beta + \omega_H, \tag{33}
\end{align*}
where $\alpha \in \wedge^{p-1}$, $\beta \in \wedge^{p+1}$ and $\omega_H$ is harmonic, $\Delta \omega_H = 0$. 
RIEMANNIAN GEOMETRY

For a metric $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$, we define a vielbein $e^a_\mu$ as a “square root” of $g_{\mu\nu}$:

$$g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab},$$  

(34)

where $\eta_{ab}$ is a local Lorentz metric. Usually, we work with positive-definite metric signature, so $\eta_{ab} = \delta_{ab}$. The inverse vielbein, which we denote by $E^\mu_a$, satisfies

$$E^\mu_a e^a_\nu = \delta^\mu_\nu; \quad E^\mu_a E^b_\mu = \delta^b_a.$$  

(35)

The “solder forms” $e^a = e^a_\mu dx^\mu$ give an orthonormal basis for the cotangent space. Similarly, the vector fields $E^\mu_a \partial_\mu$ give an orthonormal basis for the tangent space.

Torsion and curvature

We define the spin connection $\omega^a_{\ b} = \omega^a_{\ mu} dx^\mu$, the torsion 2-form $T^a$ and the curvature 2-form $\Theta^a_{\ b}$ by

$$T^a \equiv \frac{1}{2} T^a_{\ \mu\nu} dx^\mu \wedge dx^\nu = de^a + \omega^a_{\ b} \wedge e^b,$$  

(36)

$$\Theta^a_{\ b} \equiv \frac{1}{2} R^a_{\ b\mu\nu} dx^\mu \wedge dx^\nu = d\omega^a_{\ b} + \omega^a_{\ c} \wedge \omega^c_{\ b}.$$  

(37)

Define a Lorentz-covariant and general-coordinate covariant derivative $D_\mu$ that acts on tensors with coordinate and Lorentz indices:

$$D_\mu V^\nu_{\ \rho a} = \nabla_\mu V^\nu_{\ \rho a} + \omega^a_{\ mu} V^\nu_{\ \rho c} - \omega^c_{\ mu} V^\nu_{\ \rho a},$$  

(38)

where $\nabla_\mu$ is the usual general-coordinate covariant derivative:

$$\nabla_\mu V^\nu_{\ \rho} = \partial_\mu V^\nu_{\ \rho} + \Gamma^\nu_{\ \mu\sigma} V^\sigma_{\ \rho} - \Gamma^\sigma_{\ \mu\rho} V^\nu_{\ \sigma},$$  

(39)

and $\Gamma^\mu_{\ \nu\rho}$ is the Christoffel connection. Demanding metricity for $g_{\mu\nu}$, i.e. $D_\mu g_{\nu\rho} = 0$, implies

$$\Gamma^\mu_{\ \nu\rho} = \frac{1}{2} g^{\mu\sigma} \left( \partial_\nu g_{\sigma\rho} + \partial_\rho g_{\nu\sigma} - \partial_\sigma g_{\nu\rho} \right).$$  

(40)

Demanding metricity for $\eta_{ab}$, i.e. $D_\mu \eta_{ab} = 0$, implies

$$\omega_{ab} = -\omega_{ba},$$  

(41)

where $\omega_{ab} \equiv \eta_{ac} \omega^c_{\ b}$.

Bianchi Identities

Taking exterior derivative of (36) and (37) gives

$$DT^a \equiv dT^a + \omega^a_{\ b} \wedge T^b = \Theta^a_{\ b} \wedge e^b,$$  

(42)

$$D \Theta^a_{\ b} \equiv d\Theta^a_{\ b} + \omega^a_{\ c} \wedge \Theta^c_{\ b} - \Theta^a_{\ c} \wedge \omega^c_{\ b} = 0.$$  

(43)
In general, on Lorentz-valued $p$ forms such as $\alpha^{a}{}_{b}$, we define the Lorentz-covariant exterior derivative by
\[
D \alpha^{a}{}_{b} \equiv d \alpha^{a}{}_{b} + \omega^{a}{}_{c} \wedge \alpha^{c}{}_{b} - \omega^{c}{}_{b} \wedge \alpha^{a}{}_{c}.
\] (44)

**Torsion-free metric connection**

With the metricity assumption, implying (41), and the assumption that the torsion vanishes, it follows that $\omega^{a}{}_{b}$ is then uniquely determined by (36) and (41);
\[
d e^{a} = -\omega^{a}{}_{b} \wedge e^{b}; \quad \omega_{ab} = -\omega_{ba}.
\] (45)

Defining $c_{ab}{}^{c} = -c_{ba}{}^{c}$ by
\[
d e^{a} = -\frac{1}{2} c_{bc}{}^{a} e^{b} \wedge e^{c},
\] (46)

it follows that $\omega_{ab}$ is given by
\[
\omega_{ab} = \frac{1}{2} (c_{abc} + c_{acb} - c_{bca}) e^{c}.
\] (47)

Note that the vielbein is constant with respect to the Lorentz- and general-coordinate covariant derivative defined by (38); $D_{\mu} e^{a}_{\nu} = 0$.

**Symmetries of the Riemann tensor**

It follows from its definition as 2-form (37) that it is always antisymmetric on the final index pair:
\[
R_{abcd} = -R_{bacd}; \quad R_{ab\mu\nu} = -R_{ab\nu\mu},
\] (48)

where we can always freely convert coordinates indices to Lorentz indices, and *vice versa*, using the vielbein. Thus $R_{abcd} = E^{\mu}_{c} E^{\nu}_{d} R_{ab\mu\nu}$ and conversely $R_{ab\mu\nu} = e_{\mu}^{c} e_{\nu}^{d} R_{abcd}$. The metricity condition $D_{\mu} \eta_{ab} = 0$ implies $\omega_{ab} = -\omega_{ba}$, and hence $\Theta_{ab} = -\Theta_{ba}$. Thus
\[
R_{abcd} = -R_{bacd}.
\] Metricity (49)

The torsion-free condition, using (42), implies that
\[
R_{a[bc]} = 0,
\] (50)

where $R_{a[bc]} = \frac{1}{3} (R_{abcd} + R_{acdb} + R_{adbc})$. Together, (48), (49) and (50) imply
\[
R_{abcd} = R_{cdab}.
\] (51)

**The Ricci tensor and scalar, and Weyl tensor**

We define the Ricci tensor $R_{ab}$ and Ricci scalar $R$ by
\[
R_{ab} \equiv R^{c}{}_{acb}; \quad R \equiv R_{ab} \eta^{ab}.
\] (52)

Note that (51) implies that the Ricci tensor is symmetric, $R_{ab} = R_{ba}$. 

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The Weyl tensor $C_{abcd}$ is defined in $n$ dimensions by

$$C_{abcd} \equiv R_{abcd} - \frac{1}{n-2}(R_{ac} \eta_{bd} - R_{ad} \eta_{bc} + R_{bd} \eta_{ac} - R_{bc} \eta_{ad}) + \frac{1}{(n-1)(n-2)} R(\eta_{ac} \eta_{bd} - \eta_{bd} \eta_{ac}).$$  \hspace{1cm} (53)

It is the “traceless” part of the Riemann tensor, in the sense that $C^{ac}_{ac} \equiv 0$. It has the same symmetries (48)-(51) as the Riemann tensor for torsion-free connection. One may define the Weyl 2-form $\Omega_{ab}$,

$$\Omega_{ab} \equiv \frac{1}{2} C_{abcd} e^c \wedge e^d = \Theta_{ab} - \frac{1}{n-2}(R_{ac} \eta_{bd} - R_{bd} \eta_{ac}) e^c \wedge e^d + \frac{1}{(n-1)(n-2)} R\eta_{ac} \eta_{bd} e^c \wedge e^d.$$  \hspace{1cm} (54)

YANG-MILLS THEORY

If $\varphi$ is a set of scalar fields in some representation $R$ of a Lie group $G$, then we define $\varphi'$, the gauge-transformed field, by

$$\varphi' = h^{-1}\varphi,$$  \hspace{1cm} (55)

where $h = h(x)$ is a map from the base space $M$ into the group $G$. The Yang-Mills covariant derivative $D$ of $\varphi$ is defined to be

$$D\varphi \equiv (d + A)\varphi,$$  \hspace{1cm} (56)

where the Yang-Mills potential, or connection, $A$, taking its values in the adjoint representation of $G$, is defined to transform under gauge transformations as

$$A' \equiv h^{-1} Ah + h^{-1} dh.$$  \hspace{1cm} (57)

It then follows that $D\varphi$ indeed transforms in the desired covariant manner, namely

$$(D\varphi)' \equiv D'\varphi' = h^{-1} D\varphi.$$  \hspace{1cm} (58)

The Yang-Mills field strength, or curvature, $F$, is defined by

$$F \equiv dA + A \wedge A.$$  \hspace{1cm} (59)

Under gauge transformations, it transforms covariantly, as

$$F' = h^{-1} Fh.$$  \hspace{1cm} (60)

The infinitesimal forms of these transformations, when $h = 1 + \Lambda$, where $\Lambda$ is infinitesimal, reduce to the results derived in the lectures.
Kaluza-Klein and O’Neill’s formula

Given a base space $M$ with metric $ds^2$, and a principal bundle with fibre group $G$ defined over it, with connection (Yang-Mills potential) $A$, we may write down a 1-parameter family of natural metrics on the bundle as

$$d\tilde{s}^2 = \lambda^2 (\Sigma_i - A^i)^2 + ds^2,$$

(61)

where $\lambda$ is an arbitrary constant, and a summation over $i = 1, \ldots, \dim(G)$ is understood. The $\Sigma_i$ are left-invariant 1-forms on the group $G$, which means that they satisfy

$$d\Sigma_i = -\frac{1}{2} f_{ijk} \Sigma_j \wedge \Sigma_k,$$

(62)

where $f_{ijk} = f_{[ijk]}$ are the structure constants of the group. Then the Riemann tensor for the metric $d\tilde{s}^2$ is given by

$$\tilde{R}_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} - \frac{1}{4} \lambda^2 \left( F^i_{\alpha\gamma} F^i_{\beta\delta} - F^i_{\alpha\delta} F^i_{\beta\gamma} + 2 F^i_{\alpha\beta} F^i_{\gamma\delta} \right),$$

$$\tilde{R}_{\alpha\beta\gamma i} = \frac{1}{2} \lambda \partial_\gamma F^i_{\alpha\beta},$$

$$\tilde{R}_{\alpha\beta i j} = \frac{1}{4} \lambda^2 F^j_{\beta\gamma} F^i_{\alpha\gamma} - \frac{1}{4} f_{ijk} F^k_{\alpha\beta},$$

$$\tilde{R}_{ijk\ell} = \frac{1}{4\lambda^2} f_{ijm} f_{k\ell m},$$

(63)

together with those components related to the above by the Riemann tensor symmetries (48)-(51). Here we are taking the orthonormal basis

$$\tilde{e}^i = \lambda (\Sigma_i - A^i), \quad (i = 1, \ldots, \dim(G)),$$

$$\tilde{e}^{\alpha} = e^{\alpha}, \quad (\alpha = 1, \ldots, n),$$

(64)

where $e^{\alpha}$ is an orthonormal basis for the base space $M$: thus $ds^2 = e^{\alpha} e^{\alpha}$. $R_{\alpha\beta\gamma\delta}$ are the orthonormal components of the Riemann tensor on $M$, and

$$F^i = dA^i + \frac{1}{2} f_{ijk} A^j \wedge A^k,$$

$$\mathcal{D}_\gamma F^i_{\alpha\beta} = D_\gamma F^i_{\alpha\beta} + f_{ijk} A^j F^k_{\alpha\beta},$$

$$D_\gamma F^i_{\alpha\beta} = E^\mu_\gamma \left( \partial_\mu F^i_{\alpha\beta} + \omega^\alpha_{\mu\gamma} F^i_{\alpha\beta} + \omega^\beta_{\mu\gamma} F^i_{\alpha\gamma} \right).$$

(65)

(So $D_\mu$ is the Lorentz-covariant derivative, and $\mathcal{D}_\mu$ is the Lorentz and Yang-Mills covariant derivative.)