603: Electromagnetic Theory I

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1 Introduction

1.1 Maxwell's equations

The equations now known as Maxwell's equations were obtained over an extended period, principally during the early nineteenth century. Here, we shall take as our starting point the set of four differential equations as they were presented by Maxwell in about 1861. It was Maxwell who completed the process of constructing the equations, thereby achieving the first *unification* of fundamental theories in physics. Prior to Maxwell, there were two essentially independent theories, one describing electricity and the other describing magnetism, and it was he who brought about the synthesis that unified them into a single theory of *electromagnetism*. It was only later, after Einstein developed the theory of *Special Relativity* in 1905, that the magnitude of Maxwell's achievement really became clear. Especially, a quite remarkable feature of Maxell's 1861 equations is that they are already completely compatible with special relativity, with no need for modification of any kind.[1] Aside from changes in notation and units, Maxwell's equations have remained otherwise unaltered since 1861.

Let us begin by considering Maxwell's equations in free space, by which is meant that the space outside of any conducting surfaces is assumed to be a vacuum. Using the SI system of units, Maxwell's equations are:

$$\begin{aligned} \vec{\nabla}\cdot\vec{E}' &= \frac{\rho'}{\epsilon_0}\,, \qquad \vec{\nabla}\times\vec{B}' - \mu_0\epsilon_0\,\frac{\partial\vec{E}'}{\partial t} = \mu_0\vec{J}'\,,\\ \vec{\nabla}\cdot\vec{B}' &= 0\,, \qquad \vec{\nabla}\times\vec{E}' + \frac{\partial\vec{B}'}{\partial t} = 0\,. \end{aligned} \tag{1.1}$$

Observe that I have written these equations with a "prime" on the electric field $\vec{E}$ the magnetic field $\vec{B}$, the electric charge density ρ and the electric current density $\vec{J}$. This is to indicate that these quantities are all expressed in the SI system of units. The remaining quantities appearing in (1.1) are the constants ϵ_0 and μ_0, which are, respectively, the permitivity of free space and the permeability of free space. They have the values

$$\epsilon_0 \approx 8.85419\times 10^{-12} \quad \text{Farads/metre}\,, \qquad \mu_0 = 4\pi\times 10^{-7} \quad \text{Henries/metre} \tag{1.2}$$

1.2 Gaussian units

SI units have their virtues for some purposes, but they can also be quite inconvenient in practice. This seems to be especially true in electromagnetism, and for this reason it is

[1] This contrasts with the case of Newtonian mechanics, which is not compatible with special relativity and therefore did not survive as a "fundamental" theory after 1905.

often more convenient to stick with an earlier system, known as *Gaussian* units. In this system, Maxwell's equations in free space take the form

$$\begin{aligned}
\vec{\nabla}\cdot\vec{E} &= 4\pi\rho\,, \qquad \vec{\nabla}\times\vec{B} - \frac{1}{c}\,\frac{\partial\vec{E}}{\partial t} = \frac{4\pi}{c}\,\vec{J}\,,\\
\vec{\nabla}\cdot\vec{B} &= 0\,, \qquad \vec{\nabla}\times\vec{E} + \frac{1}{c}\,\frac{\partial\vec{B}}{\partial t} = 0\,,
\end{aligned} \tag{1.3}$$

where c is the speed of light. Observe that here, we are writing the equations using the unprimed quantities $\vec{E}$, $\vec{B}$, ρ and $\vec{J}$, and it will probably therefore come as no surprise that it is this Gaussian system of units that I prefer to use. It should already be evident upon comparing (1.1) and (1.3) that Gaussian system is somewhat simpler, in that one needs only one "fundamental constant" (i.e. the speed of light) rather than two (the permittivity and the permeability of free space).[2] The introduction of the 4π factors in (1.3) may perhaps seem tiresome, but the advantage of doing so will become apparent in due course. (There can be no escape from having 4π factors somewhere, because of the fact that a unit sphere has area 4π.)

In order to ensure that we can, whenever desired, revert to SI units, it is useful to work out explicitly the relation between the Gaussian quantities (denoted without primes, as in (1.3)) and the SI quantities (denoted with primes, as in (1.1)). In order to do this, we first need to understand a very important property of the Maxwell equations, namely that they imply the existence of *electromagnetic waves* that propagate at the speed of light.

Consider the Maxwell equations (1.1) in a completely empty region of space, where there is no charge density ρ' and no current density $\vec{J}'$. Taking the curl of the last equation, and using the vector identity $\vec{\nabla}\times(\vec{\nabla}\times\vec{V}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{V}) - \nabla^2\vec{V}$, we obtain

$$-\nabla^2\vec{E}' + \vec{\nabla}(\vec{\nabla}\cdot\vec{E}') + \frac{\partial}{\partial t}\vec{\nabla}\times\vec{B}' = 0\,. \tag{1.4}$$

Using the first equation in (1.1) (with $\rho' = 0$) and the second equation (with $\vec{J}' = 0$) then gives

$$\nabla^2\vec{E}' - \mu_0\epsilon_0\,\frac{\partial^2}{\partial t^2}\vec{E}' = 0\,. \tag{1.5}$$

Analogous manipulations show that $\vec{B}'$ satisfies an identical equation. We see, therefore, that the electric and magnetic fields satisfy an equation for waves that propagate at the

[2] Actually, one can do even better by changing the units in which one measures length from the *metre* to the *light second*, or alternatively, changing the unit of time to the *light metre* (the time light takes to travel 1 metre). In either of these systems of units, the speed of light becomes equal to 1.

speed

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \approx 2.99792 \times 10^8 \quad \text{metres/second}\,. \tag{1.6}$$

This is precisely the speed of light *in vacuo*, and these wave solutions describe the propagation of radio waves, light, etc.

With this preliminary, we are nearly ready to establish the relation between the SI units used in (1.1), and the Gaussian units used in (1.3). The procedure for doing this is to introduce constant factors α, β, γ and δ that relate the primed to the unprimed quantities,

$$\vec{E}' = \alpha \vec{E}\,, \quad \vec{B}' = \beta \vec{B}\,, \quad \rho' = \gamma \rho\,, \quad \vec{J}' = \delta \vec{J}\,, \tag{1.7}$$

and to fix the values of these constants by demanding that plugging (1.7) into (1.1) should give (1.3). It is essential, in doing so, that we have the relation (1.6) between c, μ_0 and ϵ_0. Elementary algebra then gives

$$\alpha = \frac{\beta}{\sqrt{\mu_0 \epsilon_0}}\,, \qquad \gamma = \delta = 4\pi\beta \sqrt{\frac{\epsilon_0}{\mu_0}}\,. \tag{1.8}$$

Observe that the value of the constant β has not yet been determined. This means that we can choose any value for β, and we may use this freedom in order to make some other equation as nice as possible by removing superfluous constant factors. Consider Coulomb's law, giving the force between two electric charges separated by a distance R. We again need to distinguish between the charges q_1' and q_2' expressed in SI units, and the charges q_1 and q_2 expressed in Gaussian units. Since we have the relation $\rho' = \gamma \rho$ between charge *densities* in the two systems, and since the unit of *volume* is the same in the two systems, it follows that the charges will also be related by the same factor of γ:

$$q' = \gamma q\,. \tag{1.9}$$

Now, in the SI system the force between the two charges is given by

$$F = \frac{q_1' q_2'}{4\pi\epsilon_0 R^2} = \frac{\gamma^2 q_1 q_2}{4\pi\epsilon_0 R^2}\,. \tag{1.10}$$

Clearly, since we are free to choose γ to be whatever we like, the sensible choice is to take

$$\gamma = \sqrt{4\pi\epsilon_0}\,, \tag{1.11}$$

so that the force between charges q_1 and q_2 is simply

$$F = \frac{q_1 q_2}{R^2}\,. \tag{1.12}$$

This is precisely the choice made in the Gaussian system of units.

Going back to (1.8), and combining it with the additional relation (1.11), we see that the four constants α, β, γ and δ are now uniquely determined in terms of μ_0 and ϵ_0. Thus we arrive at the following "dictionary" for relating the SI (primed) quantities to the Gaussian (unprimed) quantities:

$$\boxed{\begin{aligned} \vec{E}' &= \frac{1}{\sqrt{4\pi\epsilon_0}}\,\vec{E}\,, \qquad \vec{B}' = \sqrt{\frac{\mu_0}{4\pi}}\,\vec{B}\,, \\ \rho' &= \sqrt{4\pi\epsilon_0}\,\rho\,, \qquad \vec{J}' = \sqrt{4\pi\epsilon_0}\,\vec{J}\,, \qquad q' = \sqrt{4\pi\epsilon_0}\,q\,. \end{aligned}} \tag{1.13}$$

With these relations established, we can happily proceed by using the more convenient Gaussian units in this course, and anyone who wishes to re-express things in SI units can do so using the *SI-Gauss Dictionary* (1.13).

1.3 Macroscopic media

In principle, every problem in classical electromagnetism can be viewed as a problem formulated in free space, together with a number of electric point charges carried by electrons, protons, etc. In practice, however, it is often the case that the number of individual point charges is so large that it would not be convenient to consider them all separately, and instead, it is preferable to make a "macroscopic approximation." One obvious example is the notion of a conductor: It would be very clumsy and unwieldy to treat every electrostatics problem involving a conducting surface as a problem involving 10^{23} or so positive and negative point charges that are bound together in such a way as to make what we conventionally think of as a sheet of metal. Instead, we can typically just forget about the microscopic explanation of why the protons, neutrons and electrons have formed themselves into a metal, and instead simply abstract from this the macroscopic notion of a surface on which the electrostatic potential is constant.

Another example where a macroscopic viewpoint is very useful is when one considers materials (such as glass) that exhibit a dielectric permittivity, or else materials that exhibit a magnetic permeability. One certainly can give a microscopic understanding of why these materials behave as they do, but it is convenient not to have to delve into these details every time we want to work out the effect of a slab of glass in an electrostatics problem.

In order to give a macroscopic formulation of Maxwell's theory in the presence of media, we now interpret $\vec{E}$ and $\vec{B}$ as *averaged* values of the electric and magnetic fields, where the averaging is performed over the distance scale of order the interatomic spacing in the medium. The point here is that we don't want to get involved in looking at the (enormous) microscopic variations in the fields that occur on the atomic length scale as one moves around close to individual electrons and protons. Having performed this averaging, the meanings of $\vec{E}$ and $\vec{B}$ are the same as they are in free space. For example, $\vec{E}$ still measures the potential difference between neighbouring points divided by their spatial separation.

We must also introduce two new quantities, called $\vec{D}$ and $\vec{H}$, which are related to $\vec{E}$ and $\vec{B}$ respectively. The standard names for all four fields are:

$$\begin{array}{ll} \vec{E}: & \text{Electric field}\,, \\ \vec{D}: & \text{Electric displacement}\,, \\ \vec{B}: & \text{Magnetic induction}\,, \\ \vec{H}: & \text{Magnetic field}\,. \end{array}$$

In free space, we have

$$\vec{D} = \vec{E}\,, \qquad \vec{H} = \vec{B}\,. \tag{1.14}$$

In a medium, on the other hand, $\vec{D}$ represents a "back-reacted" version of $\vec{E}$, which takes into account the fact that the positive and negative charges in the medium are displaced because of the presence of the externally-applied $\vec{E}$ field, and thus they feed back into the system. To leading order, the system of positive and negative charges in the medium (which is neutral on balance) distorts so that there is an effective electric dipole, or *polarisation* $\vec{P}$, and

$$\vec{D} = \vec{E} + 4\pi\vec{P}\,. \tag{1.15}$$

In a similar way, if the medium has magnetic properties there will be a similar relation

$$\vec{H} = \vec{B} - 4\pi\vec{M}\,, \tag{1.16}$$

where $\vec{M}$ is a dipole *magnetisation* term.

The effect of all this is that the Maxwell equations are modified in the presence of the medium. Instead of the free-space equations (1.3) we shall now have

$$\boxed{\begin{aligned} \vec{\nabla}\cdot\vec{D} &= 4\pi\rho\,, \qquad \vec{\nabla}\times\vec{H} - \frac{1}{c}\frac{\partial\vec{D}}{\partial t} = \frac{4\pi}{c}\vec{J}\,, \\ \vec{\nabla}\cdot\vec{B} &= 0\,, \qquad \vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0\,, \end{aligned}} \tag{1.17}$$

Notice that it is the first two equations, the ones that have the ρ and $\vec{J}$ source-terms on the right-hand side, that are modified. The remaining two equations are completely unchanged from their free-space forms.

A common situation is when the medium is completely uniform and isotropic (meaning that it is the same in all directions), and for which $\vec{D}$ and $\vec{H}$ are simply constant multiples of $\vec{E}$ and $\vec{B}$ respectively:

$$\vec{D} = \epsilon\,\vec{E}\,, \qquad \vec{B} = \mu\,\vec{H}\,. \tag{1.18}$$

The constant ϵ is called the *relative permittivity* of the medium, and the constant μ is called the *relative permeability* of the medium. In free space, where (1.14) holds, we clearly have

$$\epsilon = 1\,, \qquad \mu = 1\,. \tag{1.19}$$

In this course, when we consider electromagnetism in a medium, we shall assume the relations (1.18).

1.4 Boundary conditions at media interfaces

A situation that one encounters frequently when studying physical problems in electromagnetism is where there is a boundary or interface between two different materials or media. The simplest such situation in electrostatics is the case where there is a conducting surface in otherwise free space. Another example would be an interface between two materials with different dialectric constants. In fact the conductor in free space can just be viewed as a special case of the interface between two dielectric materials, with one of them (free space) having $\epsilon = 1$ and the other (the conductor) having $\epsilon = \infty$.

The boundary conditions on the electric and magnetic fields at an interface between two media can be determined by performing appropriate integrals of the Maxwell equations (1.17). Let us label the media by "1" and "2," and likewise place "1" and "2" subscripts on the various electric and magnetic fields on the two sides of the interface.

Beginning with $\vec{D}$, we can integrate $\vec{\nabla}\cdot\vec{D} = 4\pi\rho$ over a so-called "Gaussian pillbox" that straddles the interface. The pillbox is a like a very short length of circular cylinder, with the ends capped off so as to form a closed surface. One should imagine that the size of the whole pillbox is very small, and in fact eventually one takes the limit where the size tends to zero. At all stages in the limiting process, the height of the box (i.e. the length of the cylinder) is very small compared with its radius. The caps of the cylinder are taken to be parallel to the interface, with the interface slicing through the box; one cap on each side.

The *divergence theorem* states that for any vector field $\vec{v}$ we have

$$\int_V \vec{\nabla}\cdot\vec{v}\,dV = \int_S \vec{v}\cdot d\vec{S}\,, \tag{1.20}$$

where S is a closed surface enclosing a volume V. Integrating $\vec{\nabla}\cdot\vec{D} = 4\pi\rho$ over the pillbox and using (1.20), we therefore find

$$\int_S \vec{D}\cdot d\vec{S} = 4\pi\int_V \rho dV = 4\pi q\,, \tag{1.21}$$

where q is the charge inside the pillbox. Because the height of the pillbox is taken to be very small compared to its diameter, we can neglect the contributions to the $\vec{D}$ integral coming from the sides. Since the pillbox itself will be taken to have infinitesimal size we can think of the interface where the pillbox is placed as being planar. Let $\vec{n}$ be the unit normal vector pointing from medium 1 into medium 2. If the cross-sectional area of the pillbox is ΔA, then (1.21) gives

$$\vec{n}\cdot(\vec{D}_2 - \vec{D}_1)\,\Delta A = 4\pi\sigma\,\Delta A\,, \tag{1.22}$$

where σ is the surface charge density (i.e. charge per unit area) at the interface. Thus we have

$$\vec{n}\cdot(\vec{D}_2 - \vec{D}_1) = 4\pi\sigma \tag{1.23}$$

at the interface.

By the same same token, the integration of the Maxwell equation $\vec{\nabla}\cdot\vec{B} = 0$ over the same pillbox gives

$$\vec{n}\cdot(\vec{B}_2 - \vec{B}_1) = 0 \tag{1.24}$$

at the interface. The zero on the right-hand side reflects the fact that there are no magnetic charges.

Further boundary conditions follow by appropriately integrating the remaining to Maxwell equations across the interface. This time, we consider a rectangular loop formed by two infinitesimally-separated parallel line elements that straddle the interface, joined by adding connecting lines at the two ends. Eventually the size of this loop is scaled to zero. We now make use of Stokes' theorem, which states that for any vector field $\vec{v}$ we have

$$\int_\Sigma (\vec{\nabla}\times\vec{v})\cdot d\vec{S} = \oint_C \vec{v}\cdot d\vec{\ell}, \tag{1.25}$$

where Σ denotes an (open) surface whose boundary is the closed loop C.

Suppose again the unit normal from medium 1 to medium 2 is $\vec{n}$ at the chosen point on the interface that we are considering. We also choose a unit vector $\vec{m}$ that is tangent to the interface at the selected point. Integrating the Maxwell equation $\vec\nabla\times\vec E = -(1/c)\partial\vec B/\partial t$ over the area Σ of the loop and applying (1.25) gives

$$\oint \vec E\cdot d\vec\ell = -\frac{1}{c}\frac{\partial}{\partial t}\int_\Sigma \vec B\cdot d\vec S\,. \tag{1.26}$$

Since $\vec B$ is assumed to be finite, as also is $\partial\vec B/\partial t$, it follows that the right-hand side is infinitesimal since the separation between the two line elements of the loop is infinitesimal, implying that the area Σ is infinitesimal. If the length of each line element in the loop is $\Delta\ell$, and if the lines are taken to be perpendicular to $\vec m$ (and, of course, also perpendicular to $\vec n$), then it follows that

$$0 = \oint \vec E\cdot d\vec\ell = (\vec m\times\vec n)\cdot(\vec E_2-\vec E_1)\Delta\ell\,. \tag{1.27}$$

Uisng the vector identity $\vec a\cdot(\vec b\times\vec c) = \vec b\cdot(\vec c\times\vec a)$ it follows that

$$\vec m\cdot[\vec n\times(\vec E_2-\vec E_1)] = 0\,. \tag{1.28}$$

This must hold for any choice of the direction for the unit tangent vector $\vec m$, and so it must be that

$$\vec n\times(\vec E_2-\vec E_1) = 0\,. \tag{1.29}$$

Finally, we perform an analogous integral over the last Maxwell equation, $\vec\nabla\times\vec H = 1/c\,\partial\vec D/\partial t + (4\pi/c)\vec J$. The finiteness of $\partial\vec D/\partial t$ means that its area integral over the loop gives zero, but $\vec J$ will have a non-zero area integral in general, since there can be a surface current density $\vec K$ (analogous to the surface charge density σ). Thus we find

$$\oint \vec H\cdot d\vec\ell = (\vec m\times\vec n)\cdot(\vec H_2-\vec H_1)\Delta\ell = \frac{4\pi}{c}\,\vec m\cdot\vec K\Delta\ell\,, \tag{1.30}$$

and since the left-hand side can be rewritten as $\vec m\cdot[\vec n\times(\vec H_2-\vec H_1)]$, and the equation must hold for all choices of direction for the tangent vector $\vec m$, we conclude that

$$\vec n\times(\vec H_2-\vec H_1) = \frac{4\pi}{c}\,\vec K\,. \tag{1.31}$$

To sumarise, the boundary conditions we have derived above in (1.23), (1.24), (1.29) and (1.31) are

$$\boxed{\begin{aligned} \vec n\cdot(\vec D_2-\vec D_1) &= 4\pi\sigma\,, \qquad \vec n\cdot(\vec B_2-\vec B_1) = 0\,,\\ \vec n\times(\vec E_2-\vec E_1) &= 0\,, \qquad \vec n\times(\vec H_2-\vec H_1) = \frac{4\pi}{c}\,\vec K\,. \end{aligned}} \tag{1.32}$$

These give the junction conditions at the interface between medium 1 and medium 2, where $\vec{n}$ is the unit normal vector pointing from 1 to 2 at the interface, σ is the surface charge density and $\vec{K}$ is the surface current density. Note that the first line of (1.32) comprises conditions on the components of the fields *normal* to the interface, whilst the second line comprises conditions on the components *parallel* to the interface.

A special case of frequent interest arises for an electric field in free space, in the presence of a conducting surface. In the free-space region we have $\vec{D} = \vec{E}$, and the conductor can be viewed as the surface of a medium having infinite dielectric constant, which means that $\vec{E} = 0$ there. Thus the pillbox integration of $\vec{\nabla} \cdot \vec{D} = 4\pi\rho$ becomes just the integral of $\vec{\nabla} \cdot \vec{E} = 4\pi\rho$, with $\vec{E} = 0$ in "medium 1." Th upshot is that the first and third junction conditions in (1.32) become (dropping the "2" subscript in the free-space region outside the conductor)

$$\vec{n} \cdot \vec{E} = 4\pi\sigma\,, \qquad \vec{n} \times \vec{E} = 0 \tag{1.33}$$

at the surface. The second equation says that there is no component of $\vec{E}$ *tangent* to the conducting surface, and the first equation says that the *normal* component of the electric field at the conductor is equal to $4\pi\sigma$.

1.5 Gauge potentials

When solving Maxwell's equations, it is often convenient to express the electric and magnetic fields in terms of potentials. This has the advantage that two of the four Maxwell equations are then explicitly solved from the outset, leaving just two more, which now become second order equations for the potentials.

Specifically, the two Maxwell equations that are solved by introducing potentials are the two that do not have ρ or $\vec{J}$ as sources, namely

$$\vec{\nabla} \cdot \vec{B} = 0\,, \qquad \vec{\nabla} \times \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0\,. \tag{1.34}$$

Notice that these two equations are exactly the same whether one is considering the free-space case (1.3) or the case where media are present (1.17).[3]

To introduce potentials we begin by considering $\vec{\nabla} \cdot \vec{B} = 0$. This can be solved by writing $\vec{B}$ as the curl of a vector:

$$\vec{B} = \vec{\nabla} \times \vec{A}\,, \tag{1.35}$$

[3]These two Maxwell equations are known as *Bianchi identities.* By contrast, the remaining two Maxwell equations are known as the *Maxwell field equations.*

since the divergence of the curl of any vector vanishes identically. Passing now to the second equation in (1.34), we plug in (1.35) and deduce (after using the fact that the partial time derivative commutes with $\vec\nabla$) that

$$\vec\nabla\times\Big(\vec E+\frac{1}{c}\,\frac{\partial\vec A}{\partial t}\Big)=0\,. \tag{1.36}$$

If a vector has vanishing curl it can be written as the gradient of a function, and so we can write $\vec E+(1/c)\partial\vec A/\partial t=-\vec\nabla\phi$.

To summarise, we can write $\vec E$ and $\vec B$ in terms of a scalar potential ϕ and a vector potential $\vec A$ as

$$\boxed{\vec E=-\vec\nabla\phi-\frac{1}{c}\,\frac{\partial\vec A}{\partial t}\,,\qquad \vec B=\vec\nabla\times\vec A\,.} \tag{1.37}$$

A feature that now arises is that the choice of potentials that give rise to given $\vec E$ and $\vec B$ fields via (1.37) is not unique. Since the curl of a gradient vanishes identically, we get the same $\vec B$ if the gradient of an arbitrary function λ is added to $\vec A$. Thus if we define

$$\vec A'=\vec A+\vec\nabla\lambda\,, \tag{1.38}$$

then $\vec A'$ gives the same $\vec B$ as does $\vec A$:

$$\vec\nabla\times\vec A'=\vec\nabla\times\vec A+\vec\nabla\times\vec\nabla\lambda=\vec\nabla\times\vec A=\vec B\,. \tag{1.39}$$

It is now evident that if, at the same time, we transform ϕ to

$$\phi'=\phi-\frac{1}{c}\,\frac{\partial\lambda}{\partial t}\,, \tag{1.40}$$

then we shall also find that ϕ' and $\vec A'$ give rise to the same $\vec E$, via (1.37), as do ϕ and $\vec A$.

To summarise, the expressions (1.37) for the electromagnetic fields give the *same* $\vec E$ and $\vec B$ if we transform the potentials ϕ and $\vec A$ according to

$$\boxed{\phi\longrightarrow\phi'=\phi-\frac{1}{c}\,\frac{\partial\lambda}{\partial t}\,,\qquad \vec A\longrightarrow\vec A'=\vec A+\vec\nabla\lambda\,,} \tag{1.41}$$

where λ is an *arbitrary* function of $\vec r$ and t. The transformations (1.41) are known as *gauge transformations*. The potentials ϕ and $\vec A$ are known as *gauge potentials*.

In this course, we shall frequently be concerned with the situation when the electric and magnetic fields are *static*, i.e. they are independent of time. It is evident from the Maxwell equations (1.3) that under these circumstances the electric and magnetic fields are totally decoupled from one another. Just because $\vec E$ and $\vec B$ are static, it does not necessarily mean that the the potentials ϕ and $\vec{A}$ have to be taken to be time independent (as can be seen by considering (1.37)). However, one would have to be quite perverse to choose to complicate a time-independent problem by opting to describe it in terms of time-dependent potentials! Thus in practice, in the static case, we always choose to take ϕ and $\vec{A}$ to be time-independent, and so (1.37) becomes simply

$$\vec{E} = -\vec{\nabla}\phi\,, \qquad \vec{B} = \vec{\nabla} \times \vec{A}\,. \tag{1.42}$$

The residual part of the gauge transformations (1.41) that preserves the time-independence of the gauge potentials is given by taking the gauge parameter λ to be of the form

$$\lambda(\vec{r}, t) = -ckt + \lambda(\vec{r})\,, \tag{1.43}$$

where k is a constant and $\lambda(\vec{r})$ is an arbitrary function of position. Thus in the static case we have independent gauge transformations under which

$$\phi \longrightarrow \phi' = \phi + k\,, \qquad \vec{A} \longrightarrow \vec{A}' = \vec{A} + \vec{\nabla}\lambda(\vec{r})\,. \tag{1.44}$$

The gauge transformation for ϕ is just the familiar freedom to add an arbitrary constant to the electrostatic potential.

1.6 Electric field of a point charge; Coulomb's law

It was found experimentally long ago, by Cavendish, Coulomb and others, that the force between two charges q_1 and q_2 in free space was proportional to the product $q_1\, q_2$; was inversely proportional to the square of the distance bewteen them (let us assume point charges, for simplicity); and was directed along the line joining the two charges. Furthermore, the force is attractive if $q_1\, q_2$ is negative, and repulsive if $q_1\, q_2$ is positive. If we work in Gaussian units, then as discussed in section 1.2, the magnitude of the force is simply equal to $q_1\, q_2$ divided by the square of the separation. All this is summarised in the equation

$$\vec{F} = q_1\, q_2\, \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^3}\,, \tag{1.45}$$

which gives the force on q_1 due to q_2, where $\vec{r}_1$ and $\vec{r}_2$ are the position vectors of the two point charges q_1 and q_2.

Coulomb also found that the force on a charge q was given by

$$\vec{F} = q\,\vec{E}\,, \tag{1.46}$$

and so we can read off from (1.45) that the electric field at the point $\vec{r}$ due to a charge q_1 located at the point $\vec{r}_1$ is given by

$$\vec{E}(\vec{r}) = q_1 \, \frac{\vec{r} - \vec{r}_1}{|\vec{r} - \vec{r}_1|^3} \,. \tag{1.47}$$

A very important and fundamental feature of electromagnetism is that it is described by a system of *linear* equations (see (1.3)), and so it obeys the principal of superposition. In particular, this means that if there are N point charges q_a located at positions $\vec{r}_a$, then the total electric field at $\vec{r}$ is simply the sum of the individual contributions from each charge:

$$\vec{E}(\vec{r}) = \sum_{a=1}^{N} q_a \, \frac{\vec{r} - \vec{r}_a}{|\vec{r} - \vec{r}_a|^3} \,. \tag{1.48}$$

We can generalise this result to the case where there is a continuum of charge, with charge density ρ. In the infinitesimal volume $dxdydz$ in the neighbourhood of the point $\vec{r}$ there will be an infinitesimal charge $\rho(\vec{r})dxdydz$. For convenience, we may write the volume element $dxdydz$ as $dxdydz \equiv d^3\vec{r}$. Then, we simply generalise the discrete sum (1.48) to an integral, and obtain

$$\vec{E}(\vec{r}) = \int \rho(\vec{r}\,') \frac{\vec{r} - \vec{r}\,'}{|\vec{r} - \vec{r}\,'|^3} \, d^3\vec{r}\,' \,. \tag{1.49}$$

The inverse process, of passing from the continuum result (1.49) to the discrete sum (1.48), can be achieved by means of the Dirac delta function. In one dimension the Dirac delta function $\delta(x - a)$ is a "spike" of zero width, infinite height, and total area=1 that has the property

$$f(a) = \int_{x_1}^{x_2} f(x)\delta(x - a)dx \tag{1.50}$$

for any function $f(x)$, provided that the integration interval $[x_1, x_2]$ encompasses the point $x = a$. (The integral would give zero if a did not lie inside the integration interval.) We then define the three-dimensional delta function $\delta^3(\vec{r} - \vec{a})$ as

$$\delta^3(\vec{r} - \vec{a}) \equiv \delta(x - a_1)\delta(y - a_2)\delta(z - a_3) \,, \tag{1.51}$$

where a_1, a_2 and a_3 are the x, y and z components of the vector $\vec{a}$: i.e. $\vec{a} = (a_1, a_2, a_3)$. Clearly, the three-dimensional delta function has the property that

$$f(\vec{a}) = \int_V f(\vec{r})\delta^3(\vec{r} - \vec{a})d^3\vec{r}, \tag{1.52}$$

provided that the integration volume V encompasses the point $\vec{r} = \vec{a}$ where the delta function has its "spike."

Using the delta function, we can then write the charge density ρ for the set of charges in (1.48) as

$$\rho(\vec r) = \sum_{a=1}^{N} q_a\, \delta^3(\vec r - \vec r_a)\,. \tag{1.53}$$

Substituting this into (1.49), and using (1.52), we indeed recover (1.48).

1.7 Gauss's law

If we are considering electrostatics, i.e. the situation where there is a time-independent electric field and no magnetic field, the Maxwell equations (1.3) in free space reduce to

$$\vec\nabla\cdot\vec E = 4\pi\rho\,, \qquad \vec\nabla\times\vec E = 0\,. \tag{1.54}$$

By integrating the first equation over a volume V, and using the divergence theorem (1.20), we obtain Gauss's law

$$\int_S \vec E\cdot d\vec S = 4\pi Q\,, \tag{1.55}$$

where S is the closed surface surrounding the volume V, and Q is the total charge contained within the volume V:

$$Q = 4\pi \int_V \rho dV\,. \tag{1.56}$$

(Actually, historically, the Maxwell equation $\vec\nabla\cdot\vec E = 4\pi\rho$ was discovered experimentally in its equivalent integrated form (1.55).)

It is instructive to examine Gauss's law in the special case of a single point charge q. Since, when we set up a Cartesian coordinate system we can choose the origin to be at any arbitrary point, it is convenient to choose it so that the charge sits at the origin. Using (1.47) we see that the electric field of the charge will be given by

$$\vec E = \frac{q\,\vec r}{r^3}\,. \tag{1.57}$$

Let us check that this is consistent with the Maxwell equation $\vec\nabla\cdot\vec E = 4\pi\rho$, and its integrated form (1.55).

First, we calculate the divergence of $\vec r/r^3$. Clearly we have $\vec\nabla\cdot\vec r = \partial x/\partial x + \partial y/\partial y + \partial z/\partial z = 3$, and since $r^2 = x^2+y^2+z^2$ we have, differentiating, $2r\partial r/\partial x = 2x$, etc., and hence $\partial r/\partial x = x/r$, etc. Thus we have

$$\vec\nabla\cdot\vec r = 3\,, \qquad \vec\nabla r = \frac{\vec r}{r}\,, \tag{1.58}$$

and so

$$\vec\nabla\cdot\Big(\frac{\vec r}{r^3}\Big) = \frac{\vec\nabla\cdot\vec r}{r^3} - \frac{3\vec r}{r^4}\cdot\frac{\vec r}{r} = \frac{3}{r^3} - \frac{3}{r^3} = 0\,. \tag{1.59}$$

This shows that for (1.57) we have in general $\vec{\nabla} \cdot \vec{E} = 0$. This is perfectly correct away from the origin, since we hope to find $\vec{\nabla} \cdot \vec{E} = 4\pi\rho$ and indeed $\rho = 0$ away from the origin. However, at the origin the calculation (1.59) is not trustworthy, because there are denominators that go to zero at $r = 0$. The safe way to handle this is to consider the integrated form of the equation, in (1.55).

Let us take the volume V in (1.55) to be a sphere of radius R centred on the origin. Plugging in (1.57), we shall then have on the left-hand side

$$q \int_S \frac{\vec{r}}{r^3} \cdot d\vec{S} = q \int_S \frac{\vec{n} \cdot d\vec{S}}{R^2} = q \int d\Omega = 4\pi q\,, \tag{1.60}$$

where we have defined the unit vector $\vec{n} \equiv \vec{r}/r$ which is the unit outward normal on the sphere of radius r, and where $d\Omega$ is the area element on the unit sphere (i.e. the solid angle element).

We obtained the result (1.60) by choosing to integrate over a sphere of radius R centred on the origin, but it is obvious that the result would be the same for *any* closed surface that surrounded the origin, and that instead we would get zero if the integration surface did not surround the origin. The conclusion from this is that the function ρ on the right-hand side of the Maxwell equation $\vec{\nabla} \cdot \vec{E} = 4\pi\rho$ must be a three-dimensional delta function centred on the origin, and that therefore the precise calculation of $\vec{\nabla} \cdot \vec{E}$ gives

$$\vec{\nabla} \cdot \vec{E} = 4\pi q\, \delta^3(\vec{r})\,. \tag{1.61}$$

1.8 Electrostatic potential

In section 1.5 we introduced gauge potentials in terms of which the electric and magnetic fields could be expressed. For the case of electrostatics, we have the particularly simple situation that the electric field is written purely in terms of a scalar potential ϕ, with

$$\vec{E} = -\vec{\nabla}\phi\,, \tag{1.62}$$

where an arbitrary constant k can clearly be added to ϕ without altering $\vec{E}$. We can see by inspection that the second Maxwell equation in (1.54) is identically satisfied when we write $\vec{E} = -\vec{\nabla}\phi$, and also that the first Maxwell equation in (1.54) becomes the Poisson equation

$$\nabla^2\phi = -4\pi\rho\,. \tag{1.63}$$

Solving any problem in electrostatics is therefore reduced to solving the Poisson equation (1.63) for a given charge density ρ, subject to given boundary conditions. This is called the *Boundary Value Problem.*

First, let us consider the potential due to a point charge q in an infinte free space. Taking the charge to be located at the origin for convenience, we therefore need to find a function ϕ such that $-\vec\nabla\phi$ is equal to $\vec E = q\,\vec r/r^3$ as in (1.57). From the second equation in (1.58) we see that $\vec\nabla(1/r) = -\vec r/r^3$, and so we may take the electrostatic potential to be

$$\phi = \frac{q}{r}\,. \tag{1.64}$$

Of course having the charge located at the origin was an inessential convenience, and we can readily write down the answer in the case where the charge is instead located at the point $\vec r_1$ just by shifting the origin of the Cartesian axes, giving

$$\phi(\vec r) = \frac{q}{|\vec r - \vec r_1|}\,. \tag{1.65}$$

Note that we can add an arbitrary constant if we wish, but in fact the choice of this "integration constant" that has been made in writing (1.65) is already rather natural, since it means that $\phi(\vec r)$ goes to zero as $\vec r$ goes to infinity.

Because of the linearity of the equations the generalisation to a system of N point charges q_a located at positions $\vec r_a$ is immediate:

$$\phi(\vec r) = \sum_{a=1}^{N} \frac{q_a}{|\vec r - \vec r_a|}\,. \tag{1.66}$$

For the case of a continuous charge distribution, we similarly have

$$\boxed{\phi(\vec r) = \int \frac{\rho(\vec r\,')}{|\vec r - \vec r\,'|}\, d^3\vec r\,'\,.} \tag{1.67}$$

These expressions are analogous to equations (1.48) and (1.49) for the electric field. Indeed, one can easily explicitly see that calculating $-\vec\nabla\phi$ for (1.66) or (1.67) gives (1.48) or (1.49) respectively. To do these calculations, one must be careful to note that $\vec\nabla$ means the gradient with respect to the coordinates in $\vec r$, and that $\vec r_a$ in (1.66), or $\vec r\,'$ in (1.67), are treated as constants in the differentiation. Thus one uses the result that

$$\vec\nabla \frac{1}{|\vec r - \vec r\,'|} = -\frac{\vec r - \vec r\,'}{|\vec r - \vec r\,'|^3}\,. \tag{1.68}$$

We had previously obtained the result (1.61) that the electric field for a point charge q at the origin, namely $\vec E = q\,\vec r/r^3$, satisfies $\vec\nabla\cdot\vec E = 4\pi q\,\delta^3(\vec r)$. Since $\vec E = -\vec\nabla\phi$ with $\phi = q/r$, we therefore have

$$\nabla^2\phi = -4\pi q\,\delta^3(\vec r)\,. \tag{1.69}$$

More generally, if the charge q is located at $\vec{r} = \vec{r}\,'$ so that the potential is given in (1.65), then we shall therefore have

$$\nabla^2 \phi = -4\pi q\, \delta^3(\vec{r} - \vec{r}\,')\,. \tag{1.70}$$

From this we can read off a result that will be useful frequently in the future, namely that

$$\boxed{\nabla^2 \frac{1}{|\vec{r} - \vec{r}\,'|} = -4\pi\, \delta^3(\vec{r} - \vec{r}\,')\,.} \tag{1.71}$$

1.9 Uniqueness theorem

Whenever one is solving a differential equation, such as the Poisson equation $\nabla^2 \phi = -4\pi\rho$ that we encounter in electrostatics, the question arises as to what boundary conditions one must impose in order to obtain a unique solution. Expressed more physically, one may ask how much boundary information must be specified in order to pin down the physics of the problem completely.

The answer for Poisson's equation is that the solution for the potential ϕ inside a volume V will be uniquely determined once its value at all points on the (closed) surface S that bounds V is specified. For example, if we are solving $\nabla^2 \phi = -4\pi\rho$ inside a sphere, then the solution will be uniquely determined provided the value of ϕ at every point on the surface of the sphere is specified. This type of boundary condition, in which ϕ is specified on S, is known as a *Dirichlet boundary condition.* An alternative is to specify not ϕ itself, but its *normal derivative* $\partial\phi/\partial n$, on the boundary. This is known as a *Neumann boundary condition.* In this case the solution for ϕ is again unique, except for the (trivial) point that an arbitrary additive constant is undetermined. One can also consider mixed boundary conditions, which are Dirichlet on parts of the boundary and Neumann on the rest.

To prove these statements, we suppose that for given boundary conditions on S there exist two different solutions to $\nabla^2 \phi = -4\pi\rho$. Let these solutions be ϕ_1 and ϕ_2. The idea will be to prove that actually $\phi_1 = \phi_2$, and so the solution is unique. With

$$\nabla^2 \phi_1 = -4\pi\rho\,, \qquad \nabla^2 \phi_2 = -4\pi\rho\,, \tag{1.72}$$

it follows by subtraction that the function ψ defined by

$$\psi \equiv \phi_1 - \phi_2 \tag{1.73}$$

will satisfy Laplace's equation

$$\nabla^2 \psi = 0 \tag{1.74}$$

in the volume V. Since ϕ_1 and ϕ_2 by definition satisfy identical boundary conditions on S, it follows that ψ will satisfy either $\psi = 0$ (Dirichlet) or $\partial\psi/\partial n = 0$ (Neumann) on S.

We now multiply (1.74) by ψ, integrate over V, and then perform an integration by parts:

$$
\begin{aligned}
0 &= \int_V \psi \nabla^2 \psi \, dV \,, \\
&= \int_V \Big[\vec\nabla \cdot (\psi \vec\nabla \psi) - \vec\nabla\psi \cdot \vec\nabla\psi\Big] dV \,, \\
&= \int_S \psi \vec\nabla\psi \cdot d\vec S - \int_V |\vec\nabla\psi|^2 \, dV \,.
\end{aligned}
\tag{1.75}
$$

(The first term on the last line comes by using the divergence theorem (1.20).) Now we see that since either ψ or its normal derivative $\partial\psi/\partial n \equiv \vec n \cdot \vec\nabla\psi$ vanishes at all points on S, we are left with

$$\int_V |\vec\nabla\psi|^2 \, dV = 0 \,. \tag{1.76}$$

The integrand is everywhere non-negative, and so the integral can only be zero if the integrand vanishes everywhere in V. But if $|\vec\nabla\psi|^2{=}0$ it follows that

$$\vec\nabla\psi = 0 \tag{1.77}$$

everywhere in V, and so we conclude that $\psi =$constant everywhere in V. In other words, we have proved that

$$\phi_1 = \phi_2 + k \,, \tag{1.78}$$

where k is a constant.

In the case of Dirichlet boundary conditions we know that $\phi_1 = \phi_2$ on S, and so the constant k must be zero. This proves that $\phi_1 = \phi_2$ everywhere in V, thus establishing that the solution is unique.

In the case of Neumann boundary conditions, where only the normal derivative is specified on S, it is clear that the constant k can never be determined. This is of no consequence, since ϕ and $\phi + k$ give rise to the same physical $\vec E$ field in any case. So in the Neumann case too, the solution is unique.

Note that the results above can apply not only to the problem of solving for ϕ inside a finite volume V with finite-sized closed boundary S, but also in the case where volume V is infinite. A typical example would be when there is a finite-sized surface S_1 (for example a spherical conductor) and the volume V is taken to be the entire infinite space outside it. In this case there is no actual boundary at infinity, but we can treat the problem by imagining that we introduce a spherical boundary surface S_2 at some very large radius R, and eventually we send R to infinity. When R is large but finite, we have a finite volume V bounded by the disconnected sum of the two surfaces S_1 (in the middle) and S_2 (at large distance).

The uniqueness arguments discussed above can then be applied to this situation, with the surface integral in (1.75) becoming the sum of two integrals, one over the component S_1 of the total boundary and the other over the component S_2. Dirichlet or Neumann boundary conditions are specified on S_1, and so that contribution to the surface integral will vanish. The surface integral over S_2 will become zero in the limit that the radius R is sent to infinity, provided that ϕ goes to zero sufficiently fast at infinity. Thus in practice we think of S_2 as "the sphere at infinity," and we impose the boundary condition that ϕ goes to zero at infinity, thereby ensuring that the S_2 component of the surface integral in (1.75) will vanish too. This ensures that again we are left with just the volume integral (1.76), and so the uniqueness proof goes through as before.

Note also that we can allow multiple disconnected surfaces at finite distance, provided that Dirichlet or Neumann boundary conditions are imposed on all of them.

In summary, therefore, we have a uniqueness proof too in the case where the volume V is infinite, provided that we not only impose Dirichlet or Neumann boundary conditions on any boundary surfaces at finite distance, but we also impose a fall-off condition on the potential at infinity.

Note that we have established uniqueness of the solution subject to the imposition of *either* Dirichlet *or* Neumann boundary conditions at each point on the boundary. It could be Dirichlet for some points, and Neumann for others, but at any given point one must specify only one of Dirichlet or Neumann. With such boundary conditions specified, the problem is said to be *well posed*. This means that these boundary conditions are neither too weak, leaving the problem underdetermined and not fully pinned down, nor are they too strong, leaving the problem overdetermined and therefore admitting no solution.

An example of an overdetermined problem would be if one tried to impose *both* Dirichlet *and* Neumann boundary conditions at each point on S. In other words, if one tried to specify both the potential *and* its normal derivative at each point on S. Specifying both ϕ and $\partial\phi/\partial n$ on S is known as specifying *Cauchy* boundary conditions. That this would be an overdetermination is obvious from the fact that Dirichlet conditions alone are sufficient to give a unique solution. And, on the other hand, Neumann conditions alone are sufficient to give another unique solution. Except in the unlikely event that one picked precisely the "matching" set of Neumann conditions that would reproduce the solution with the Dirichlet conditions, there will be a conflict between the two, implying that no solution would exist.

1.10 Green's theorem

In section 1.8 we gave the expression (1.67) for the electrostatic potential due to a distribution of charge with charge density ρ:

$$\phi(\vec{r}) = \int \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\, d^3\vec{r}\,' \tag{1.79}$$

This result assumes that the charge distribution exists in otherwise free space, with no conductors or other boundaries present. In practice, as we have already remarked, a typical realistic situation is one where there are other conductors, etc., on which boundary conditions are specified.

To handle the case where there are boundaries, the following procedure can be useful. We first derive a simple result known as *Green's theorem*, and then apply it to the case of interest.

Let ϕ and ψ be two scalar functions. We can then consider

$$\begin{aligned} \vec{\nabla}\cdot(\phi\vec{\nabla}\psi - \psi\vec{\nabla}\phi) &= \vec{\nabla}\phi\cdot\vec{\nabla}\psi + \phi\nabla^2\psi - \vec{\nabla}\psi\cdot\vec{\nabla}\phi - \psi\nabla^2\phi\,,\\ &= \phi\nabla^2\psi - \psi\nabla^2\phi\,. \end{aligned} \tag{1.80}$$

Integrating this over a volume V bounded by surface S, and using the divergence theorem (1.20), we therefore find

$$\boxed{\int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)dV = \int_S (\phi\vec{\nabla}\psi - \psi\vec{\nabla}\phi)\cdot d\vec{S}\,.} \tag{1.81}$$

This is Green's theorem.

We apply it to our electrostatics problem by taking ϕ to be the electrostatic potential satisfying Poisson's equation (1.63), and taking

$$\psi = \frac{1}{|\vec{r}-\vec{r}\,'|}\,. \tag{1.82}$$

We shall take $\vec{r}\,'$ to be the integration variable in (1.81), and so the derivatives in (1.81) will also be with respect to $\vec{r}\,'$. We shall therefore denote these with primes also. Note from (1.71) that we shall have

$$\nabla'^2 \frac{1}{|\vec{r}-\vec{r}\,'|} = -4\pi\delta^3(\vec{r}\,'-\vec{r})\,. \tag{1.83}$$

This can be used on the left-hand side of (1.81), and we also use $\nabla'^2\phi(\vec r\,') = -4\pi\rho(\vec r\,')$. Thus we obtain

$$\boxed{\phi(\vec r) = \int_V \frac{\rho(\vec r\,')}{|\vec r - \vec r\,'|}\, d^3\vec r\,' + \frac{1}{4\pi}\int_S \Big[\frac{1}{|\vec r - \vec r\,'|}\,\vec\nabla'\phi(\vec r\,') - \phi(\vec r\,')\,\vec\nabla'\Big(\frac{1}{|\vec r - \vec r\,'|}\Big)\Big]\cdot d\vec S' \,. \qquad (1.84)}$$

The first term on the right-hand side of (1.84) is of the same form as the expression (1.79) that held in free space. The surface integrals in (1.84) represent the contribution from charge distributions on the boundary S that we are now including.

We can interpret (1.84) as giving the expression for the potential everywhere in V in terms of the charge density ρ in V and the potential (and its normal derivative) on S. However, we cannot view (1.84) in itself as providing the answer we are seeking for how to solve for the potential in a general electrostatics problem. It can be seen from (1.84) that we would need to feed in the information about ϕ on the boundary *and also* about $\partial\phi/\partial n$ on the boundary in order to obtain the expression for ϕ in V. But we saw in the discussion of the uniqueness theorem that we are not allowed to specify independently the values of ϕ and also of its normal derivative on the boundary; that would give an overdetermined problem that admitted no solution.

Thus we can only regard (1.84) as an integral equation which will tell us what ϕ is everywhere, once we know what it and its normal derivative are on the boundary. To solve the general boundary-value problem we will need to introduce another tool, which is called the *Green function.*

1.11 Green functions and the boundary-value problem

Although (1.84) solves the desired Poisson equation (1.63) with the desired boundary conditions, it is unsatisfactory as a solution of the boundary-value problem because one would have to know both ϕ and its normal derivative on the boundary, whereas in fact these are not independent pieces of information and so they cannot be independently specified. The difficulty would be overcome if we could somehow arrange that only one of the two surface-integral terms in (1.84) were present. This can be achieved by changing the choice for the function ψ in (1.82) that we inserted into Green's theorem (1.81) in order to obtain (1.84). Instead of taking ψ to be simply given by (1.82), we need to be a little more ingenious. The function we require is known as a *Green function.*[4]

[4] One sometimes sees this referred to as "a Green's function." This is a grammatical abomination, which is as wrong as calling an apple pie "an apple's pie." There are numerous offenders in the physics community.

The key point about the function $\psi = |\vec{r} = \vec{r}\,'|^{-1}$ that we needed in deriving the result (1.84) was that it satisfied (1.83). In fact there is a much broader class of functions that satisfy equation (1.83). This is an inhomogeneous equation with the delta function providing a source on the right-hand side, and so we can add to the solution (1.82) an arbitrary solution of the homogeneous equation. Thus we may take ψ in (1.81) to be any function of the form

$$G(\vec{r},\vec{r}\,') = \frac{1}{|\vec{r}-\vec{r}\,'|} + F(\vec{r},\vec{r}\,')\,, \tag{1.85}$$

where $F(\vec{r},\vec{r}\,')$ is any solution of the homogeneous equation

$$\nabla'^2 F(\vec{r},\vec{r}\,') = 0\,. \tag{1.86}$$

(i.e. $F(\vec{r},\vec{r}\,')$ is an *harmonic* function.) Thus we have

$$\nabla'^2 G(\vec{r},\vec{r}\,') = -4\pi\delta^3(\vec{r}-\vec{r}\,')\,. \tag{1.87}$$

The idea now is that we will *choose* $F(\vec{r},\vec{r}\,')$ so that $G(\vec{r},\vec{r}\,')$, which is called a *Green function*, satisfies appropriate boundary conditions.

To see how this works, we first note that there is an analogous result to (1.84) where we take ψ in (1.81) to be $G(\vec{r},\vec{r}\,')$ rather than $|\vec{r}-\vec{r}\,'|^{-1}$, namely

$$\phi(\vec{r}) = \int_V \rho(\vec{r}\,')\,G(\vec{r},\vec{r}\,')d^3\vec{r}\,' + \frac{1}{4\pi}\int_S \Big[G(\vec{r},\vec{r}\,')\,\vec{\nabla}'\phi(\vec{r}\,') - \phi(\vec{r}\,')\,\vec{\nabla}'G(\vec{r},\vec{r}\,')\Big]\cdot d\vec{S}'\,. \tag{1.88}$$

Consider first the case where we wish to specify Dirichlet boundary conditions for the potential on the surface S. We achieve this by choosing the harmonic function $F(\vec{r},\vec{r}\,')$ in (1.85) so that $G(\vec{r},\vec{r}\,')$ vanishes when $\vec{r}\,'$ lies in the surface S. Thus, denoting this *Dirichlet Green function* by $G_D(\vec{r},\vec{r}\,')$, we have

$$G_D(\vec{r},\vec{r}\,') = 0 \qquad \text{when} \quad \vec{r}\,' \in S\,. \tag{1.89}$$

Using $G_D(\vec{r},\vec{r}\,')$ in (1.88) we therefore obtain

$$\boxed{\phi(\vec{r}) = \int_V \rho(\vec{r}\,')\,G_D(\vec{r},\vec{r}\,')d^3\vec{r}\,' - \frac{1}{4\pi}\int_S \phi(\vec{r}\,')\,\vec{\nabla}'G_D(\vec{r},\vec{r}\,')\cdot d\vec{S}'\,. \qquad (1.90)}$$

This has achieved the goal of giving an expression for $\phi(\vec{r})$ everywhere in the volume V, expressed in terms of the given charge density $\rho(\vec{r})$ and the values of $\phi(\vec{r})$ on the boundary surface S. Thus, we may say that the Dirichlet boundary-value problem is solved, albeit somwhat formally.

J.D. Jackson used to be one of them, but although he referred to "a Green's function" in the first edition of *Classical Electrodynamics*, he had reformed by the time the second edition appeared.

One might worry that (1.90) has done little more than replace one difficult problem (solving $\nabla^2\phi = 4\pi\rho$ for ϕ) by another equivalently difficult problem (solving for the harmonic function $F(\vec r, \vec r\,')$ that is needed in order to ensure the Green function satisfies (1.89)). However, this is not quite true, and moreover, a very important advantage of adopting this Green-function approach is that solving just once for the Green function for the given geometry then allows us to solve many different boundary-value problems.

The point is the following. Having once solved for $G_D(\vec r, \vec r\,')$ in the given geometry (i.e. for the specified boundary surface S), one can now construct the solution for $\phi(\vec r)$ for *any* choice of charge density $\rho(\vec r)$ and for *any* choice of the boundary-value potential on the surface S. Thus finding the Dirichlet Green function just once for the chosen geometry allows us to solve *any* Dirichlet boundary-value problem for that geometry.

The solution to the Neumann problem goes rather similarly, although with a minor subtlety. One might think that now one should choose $F(\vec r, \vec r\,')$ in (1.85) so that the normal derivative of the Green function vanished on S,

$$\frac{\partial G_N(\vec r, \vec r\,')}{\partial n'} = 0 \qquad \text{when} \quad \vec r\,' \in S\,. \tag{1.91}$$

But this would lead to a contradiction, since by integrating (1.87) over V and using the divergence theorem (1.20) gives

$$\int_S \vec\nabla'\, G_N(\vec r, \vec r\,') \cdot d\vec S' = -4\pi\,, \tag{1.92}$$

and so we cannot impose the boundary condition (1.91) on S. The simplest choice is to impose

$$\frac{\partial G_N(\vec r, \vec r\,')}{\partial n'} = -\frac{4\pi}{A} \qquad \text{when} \quad \vec r\,' \in S\,, \tag{1.93}$$

where A is the area of the boundary S.

Substituting (1.93) into (1.88), we therefore find

$$\boxed{\phi(\vec r) = \langle\phi\rangle_S + \int_V \rho(\vec r\,')\, G_N(\vec r, \vec r\,') d^3\vec r\,' + \frac{1}{4\pi}\int_S G_N(\vec r, \vec r\,')\, \vec\nabla'\phi(\vec r\,') \cdot d\vec S'\,, \qquad (1.94)}$$

where $\langle\phi\rangle_S$ denotes the average value of ϕ over the surface S,

$$\langle\phi\rangle_S = \frac{1}{A}\int_S \phi dS\,. \tag{1.95}$$

The solutions (1.90) or (1.94) for ϕ in terms of the Dirichlet or Neumann Green function provide at least a formal solution to the boundary-value problem. How useful they are in practice depends upon the details of the geometry of the problem. It all comes down to the question of whether one can solve explicitly for the Green function $G_D(\vec r, \vec r\,')$ or $G_N(\vec r, \vec r\,')$. For a boundary S of some generic type it will certainly be impossible. In certain special cases one can obtain closed-form expressions. We shall meet examples later on where this can be done, in the case of an infinite planar boundary S, and in the case of a spherical boundary.

It is worth making a few closing remarks about the physical interpretation of the Green function. The simplest example is when

$$G(\vec r, \vec r\,') = \frac{1}{|\vec r - \vec r\,'|}\,, \tag{1.96}$$

which is, as we have seen, the Green function for the Dirichlet problem where the only boundary is the "sphere at infinity." We can recognise (1.96) as being the electric potential at the point $\vec r$ due to a unit point charge at the point $\vec r\,'$. In fact (1.96) is symmetrical under the exchange of $\vec r$ and $\vec r\,'$, and so it can be equivalently viewed as the potential at $\vec r\,'$ due to a unit charge at $\vec r$.

In the more general case, the Green function is of the form (1.85), where $F(\vec r, \vec r\,')$ is harmonic function. This means that $G(\vec r, \vec r\,')$ again has the interpretation of being the potential at $\vec r$ due to a unit charge at $\vec r\,'$, but now in the more complicated case where $G(\vec r, \vec r\,')$ (or its normal derivative, in the Neumann case) vanishes on S.

One can show quite generally that in the case of Dirichlet boundary conditions, the Green function $G_D(\vec r, \vec r\,')$ is necessarily symmetrical under the exchange of $\vec r$ and $\vec r\,'$:

$$G_D(\vec r, \vec r\,') = G_D(\vec r\,', \vec r)\,. \tag{1.97}$$

(This can be done by using Green's theorem (1.81).) In the case of Neumann boundary conditions, symmetry under the exchange of $\vec r$ and $\vec r\,'$ is not automatic, but it can always be imposed.

1.12 Electrostatic energy

The force on a charge q in an electric field $\vec E$ is given by $\vec F = q\vec E$. It follows that the work done in moving it from a point P_1 to a point P_2 is given by

$$\Delta W = -\int_{P_1}^{P_2} \vec F \cdot d\vec\ell = -q \int_{P_1}^{P_2} \vec E \cdot d\vec\ell\,. \tag{1.98}$$

Using $\vec E = -\vec\nabla\phi$ therefore gives

$$\Delta W = q \int_{P_1}^{P_2} \vec\nabla\phi \cdot d\vec\ell = q\phi(P_2) - q\phi(P_1)\,. \tag{1.99}$$

From this, we can read off the potential energy U of the charge q in the electrostatic field as being

$$\boxed{U = q\,\phi\,.} \tag{1.100}$$

If we consider a system of N charges q_a at point $\vec{r}_a$ in free space, then the the electrostatic energy of a particular charge q_a in to the potential due to the other $N-1$ charges will therefore be

$$U_a = q_a \sum_{b\neq a} \frac{q_b}{|\vec{r}_a - \vec{r}_b|}\,, \tag{1.101}$$

where the sum is taken over the remaining $N-1$ charges. The total potential energy will then be given by

$$U = \sum_{a=1}^{N}\sum_{b<a} \frac{q_a q_b}{|\vec{r}_a - \vec{r}_b|}\,. \tag{1.102}$$

(The second summation is over $b < a$ rather than over all $b \neq a$ to avoid a double counting: the energy of charge 1 in the field of charge 2 is the same as the energy of charge 2 in the field of charge 1, but this energy counts *only once* to the total energy, not twice.) Another way of writing this is

$$U = \tfrac{1}{2}\sum_{a}\sum_{b\neq a} \frac{q_a q_b}{|\vec{r}_a - \vec{r}_b|}\,, \tag{1.103}$$

where it is understood that, aside from the exclusion $b \neq a$, the indices a and b range over 1 to N. The exclusion of $b = a$ is necessary since otherwise the formula (1.103) would give a nonsensical infinite result. The excluded terms can be thought of as the infinite "self-energy" contributions.

Generalising (1.103) to the case of a continuous charge distribution clearly gives

$$U = \tfrac{1}{2}\int\!\!\int \frac{\rho(\vec{r})\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|}\, d^3\vec{r}\, d^3\vec{r}\,'\,. \tag{1.104}$$

In view of the fact that the potential $\phi(\vec{r})$ is given by (1.67), we can rewrite (1.104) as

$$U = \tfrac{1}{2}\int \rho(\vec{r})\phi(\vec{r}) d^3\vec{r}\,. \tag{1.105}$$

We can also rewrite the energy purely in terms of the electric field. To do this, we use the Poisson equation (1.63) to write ρ in (1.105) as $\rho = -1/(4\pi)\,\nabla^2\phi$, giving

$$\begin{aligned} U &= -\frac{1}{8\pi}\int \phi\nabla^2\phi d^3\vec{r}\,,\\ &= -\frac{1}{8\pi}\int \vec{\nabla}\cdot(\phi\vec{\nabla}\phi) d^3\vec{r} + \frac{1}{8\pi}\int |\vec{\nabla}\phi|^2\, d^3\vec{r}\,,\\ &= -\frac{1}{8\pi}\int_S \phi\vec{\nabla}\phi\cdot d\vec{S} + \frac{1}{8\pi}\int |\vec{\nabla}\phi|^2\, d^3\vec{r}\,,\\ &= \frac{1}{8\pi}\int |\vec{\nabla}\phi|^2\, d^3\vec{r}\,. \end{aligned} \tag{1.106}$$

Note that after performing the integration by parts here, we have dropped the surface term coming from the "sphere at infinity" since the fields are assumed to go to zero there.

Since $\vec{E} = -\vec{\nabla}\phi$ we therefore have

$$U = \frac{1}{8\pi} \int |\vec{E}|^2 d^3\vec{r}\,, \tag{1.107}$$

integrated over all space. This leads naturally to the definition of *energy density* in the electric field as

$$w = \frac{1}{8\pi} |\vec{E}|^2\,. \tag{1.108}$$

It is of interest to apply this result to the electric field at the surface of a conductor. Recall from section 1.4 that we found that the electric field is always normal to the surface of a conductor, and that there is a surface-charge density σ given by $\vec{n}\cdot\vec{E} = 4\pi\sigma$ (see(1.33)). Therefore, in the neighbourhood of the conductor there is an energy density

$$w = \frac{1}{8\pi} |\vec{E}|^2 = 2\pi\sigma^2\,. \tag{1.109}$$

If an element of the surface of area δA is displaced outwards by a distance δx, the electrostatic energy will then change by an amount

$$\delta U = -w\delta A\,\delta x = -2\pi\sigma^2\delta A\,\delta x\,. \tag{1.110}$$

This implies an outward force per unit area (i.e. pressure) given by

$$p = 2\pi\sigma^2\,. \tag{1.111}$$

2 Method of Images

Sometimes, if there is a boundary surface of particularly simple and symmetrical geometry, it is possible to solve the boundary-value problem by means of the *Method of Images*. Suppose, for example, the potential is specified to be zero on a surface S in otherwise free space, and that one wishes to calculate the potential everywhere outside (in the volume V) due to a point charge located outside the surface. If S is suitably symmetrical, it may be possible to "mock up" the same zero-potential surface by considering a totally free space, with no surfaces anywhere, but with one or more additional *image charges* judiciously introduced in the region of the total space that does *not* include V. (i.e. , the additional image charges are on the "other side" of where the surface S of the original problem was located.) Suppose that by introducing image charges in this way, one can arrange that the total potential due to the original charge plus the image charges is zero on the "virtual" surface S. It is then clear, by invoking the uniqueness theorem (see section 1.9), that the potential at all points in V must be the same in the image-charge "mock-up" and in the original problem with the actual physical conducting surface.

The challenge is to figure out how to achieve the "virtual" zero-potential surface by means of image charges. In practice, there are very few cases where it can be done. We shall discuss two of them now.

2.1 Infinite planar conductor

The simplest example where the method of images can be employed is in the case of an infinite planar conductor. Let us suppose, for convenience, that Cartesian axes are chosen so that the conductor lies in the plane $z = 0$. We shall take the volume V that lies "outside" the conductor to be the whole of the half-space $z > 0$. Suppose the conductor is fixed at zero potential.

If a point charge q is located at some point in V, then it is obvious that if an image charge $-q$ is placed "behind" the conductor at precisely the mirror-image location, then by symmetry it must be the case that the total potential of original plus image charge, now taken to be in a completely free space with no conductor at all, will vanish on the plane $z = 0$. Therefore, the potential at any point in V in the original problem with conductor will be given by the total potential in the image-charge "mock up."

To be more precise, let us suppose that the original charge q is located at

$$\vec{r}_1 = (x_1, y_1, z_2)\,, \qquad z > 0\,. \tag{2.1}$$

The image charge $-q$ will then be located at

$$\vec{r}_2 = (x_2, y_2, z_2) = (x_1, y_1, -z_1)\,. \tag{2.2}$$

Therefore, the total potential is given by

$$\begin{aligned} \phi(\vec{r}) &= \frac{q}{|\vec{r}-\vec{r}_1|} - \frac{q}{|\vec{r}-\vec{r}_2|}\,, \qquad (2.3)\\ &= \frac{q}{\sqrt{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2}} - \frac{q}{\sqrt{(x-x_1)^2+(y-y_1)^2+(z+z_1)^2}}\,. \end{aligned}$$

Clearly this potential indeed vanishes on the surface $z = 0$, and so therefore by the uniqueness theorem $\phi(\vec{r})$ describes the potential, at all points with $z > 0$, of the single charge q in the presence of the infinite conductor at $z = 0$.

2.2 Dirichlet and Neumann Green functions for infinite planar boundary

We are now in a position to construct the Dirichlet and Neumann Green functions for this case. Recall that the Dirichlet Green function $G_D(\vec{r},\vec{r}\,')$ is defined to be the potential at $\vec{r}$ due to a unit strength charge at $\vec{r}\,'$, subject to the condition that $G_D(\vec{r},\vec{r}\,')$ should vanish on the boundary S. Thus we can read off from (2.4) that the Dirichlet Green function in the case of the infinite planar boundary at $z = 0$ is given by

$$G_D(x,y,z;x',y',z') = \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} - \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}}\,. \tag{2.4}$$

Note that, as asserted in section 1.11, this is indeed symmetric under the exchange of $\vec{r}$ and $\vec{r}\,'$.

To use (2.4) in order to solve Dirichlet boundary-value problems in this geometry, we just plug it into the general expression (1.90). This requires that we evaluate the normal derivative of G_D on the boundary, which in the present case means that we need

$$-\frac{\partial G_D(x,y,z;x',y',z')}{\partial z'}\Big|_{z'=0} = -\frac{2z}{[(x-x')^2+(y-y')^2+z^2]^{3/2}}\,. \tag{2.5}$$

(The normal derivative in (1.90) is directed *outwards* from the volume V, which means in this case in the *negative* direction along the z' axis.) Suppose, for simplicity, we consider the situation where there is no charge distribution ρ in the problem, and so the entire contribution to the potential $\phi(\vec{r})$ comes from the boundary contribution in (1.90). Then we find

$$\phi(x,y,z) = \frac{z}{2\pi}\int \frac{\phi(x',y',0)\,dx'dy'}{[(x-x')^2+(y-y')^2+z^2]^{3/2}}\,. \tag{2.6}$$

Thus we have constructed the solution of the source-free equation $\nabla^2\phi = -4\pi\rho = 0$, in which the boundary value of ϕ on the infinite planar surface $z' = 0$ is specified.

We can also easily construct the Neumann Green function $G_N(\vec{r},\vec{r}\,')$ for this geometry. In this case, it is defined to be the potential at $\vec{r}$ due to a unit strength charge at $\vec{r}\,'$, subject to the condition that the normal derivative of G_N should vanish on the plane $z' = 0$.[5] This time, we can suspect that a small modification of the image-charge trick should give us the required result. Indeed this works, and all we need to do is to replace the minus sign in

[5] In our general discussion for the Green function for Neumann boundary conditions, we had the requirement (1.93) that the normal derivative should equal $-\pi/A$, where A was the area of the boundary. In the present case this area is infinite, and so we can simply require that the normal derivative of G_N should vanish.

front of the second term in (2.4) by a plus sign, to give

$$G_N(x,y,z;x',y',z') = \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} + \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}}\,. \tag{2.7}$$

It is easily seen that this satisfies the required condition that

$$\frac{\partial G_N(x,y,z;x',y',z')}{\partial z'}\Big|_{z'=0} = 0\,. \tag{2.8}$$

We now plug the Neumann Green function into (1.94), in order to solve the general class of boundary-value problems in which the normal derivative of ϕ is specified on the infinite planar surface $z=0$. Suppose again, for simplicity, we consider the case where $\rho=0$. Plugging (2.7) into (1.94) then gives

$$\phi(x,y,z) = -\frac{1}{2\pi}\int \frac{1}{\sqrt{(x-x')^2+(y-y')^2+z^2}}\,\Big(\frac{\partial\phi(x',y',z')}{\partial z'}\Big|_{z'=0}\Big)\,dx'dy'\,. \tag{2.9}$$

(The minus sign arises because the outward normal derivative at $z'=0$ is $-\partial/\partial z'$.) Of course, since $\vec{E} = -\vec{\nabla}\phi$, we may write (2.10) as

$$\phi(x,y,z) = \frac{1}{2\pi}\int \frac{E_z(x',y',0)\,dx'dy'}{\sqrt{(x-x')^2+(y-y')^2+z^2}}\,. \tag{2.10}$$

2.3 Spherical conductor

A slightly more subtle example where the method of images can be employed is in the case of a spherical conductor. Suppose a conducting sphere of radius a is held at zero potential, and that a charge q is placed outside the sphere, at a distance b from its origin. It turns out that this situation can be "mocked up" by considering instead entirely free space containing the original charge and also a certain charge q' placed at a certain distance c from the origin of the sphere, on the line joining the charge q and the origin.

The quickest way to derive this result is as follows. Imagine that the sphere is centred on the origin of Cartesian coordinates, and that the charge q is placed at distance b along the z axis, i.e. at $(x,y,z)=(0,0,b)$. The claim is that the image charge q' should also lie on the z axis, at some point $(x,y,z)=(0,0,c)$. If this does indeed give rise to a spherical surface of radius a that has zero potential, then in particular it must be that the potential is zero at the two points $(0,0,a)$ and $(0,0,-a)$ on the sphere. Since these two points are aligned on the same axis as the charges, it is particularly easy to write down the conditions that the potential should be zero:

$$\frac{q}{b+a} + \frac{q'}{a+c} = 0\,, \qquad \frac{q}{b-a} + \frac{q'}{a-c} = 0\,. \tag{2.11}$$

These two conditions determine q' and c, giving

$$q' = -\frac{aq}{b}\,, \qquad c = \frac{a^2}{b}\,. \tag{2.12}$$

Observe that since $b > a$, we have $c = a(a/b) < a$. Thus, as one would expect, the image charge is *inside* the spherical surface. It remains to verify that the potential then vanishes for an arbitrary point on the sphere. The problem has rotational symmetry around the z axis, so it suffices to consider a point P at angle θ from the z-axis. If the distance from q to P is ℓ_q, and the distance from q' to P is $\ell_{q'}$, then the cosine rule gives

$$\ell_q^2 = a^2 + b^2 - 2ab\cos\theta\,, \qquad \ell_{q'}^2 = a^2 + c^2 - 2ac\cos\theta\,. \tag{2.13}$$

After using (2.12) we see that $\ell_{q'} = (a/b)\,\ell_q$ and that indeed we have

$$\frac{q}{\ell_q} + \frac{q'}{\ell_{q'}} = 0 \tag{2.14}$$

for all θ, and so the potential vanishes everywhere on the sphere $x^2 + y^2 + z^2 = a^2$.

It is useful also to give the result in a more general fashion, in which the original charge q is placed at an arbitrary point $\vec r_1$ located outside the sphere, rather than lying specifically on the z axis.

Clearly, if the charge q lies at $\vec r_1$ then the charge q' must lie at a point $\vec r_2$ along the same direction, and since the second relation in (2.4) can be written as $c = (a^2/b^2)\,b$, we must have

$$\vec r_2 = \frac{a^2}{{r_1}^2}\,\vec r_1\,. \tag{2.15}$$

Thus, the potential at $\vec r$ outside the zero-potential sphere at $r = a$ due to a charge q located at $\vec r_1$ outside the sphere is given by

$$\phi(\vec r) = \frac{q}{|\vec r - \vec r_1|} - \frac{qa/r_1}{|\vec r - (a^2/r_1^2)\,\vec r_1|}\,. \tag{2.16}$$

If we define γ to be the angle between $\vec r$ and $\vec r_1$, so that $\vec r\cdot\vec r_1 = rr_1\,\cos\gamma$, then (2.16) can be expressed as

$$\phi(\vec r) = \frac{q}{(r^2 + r_1^2 - 2rr_1\,\cos\gamma)^{1/2}} - \frac{qa}{(r^2 r_1^2 + a^4 - 2a^2\,rr_1\cos\gamma)^{1/2}}\,. \tag{2.17}$$

Using the expression for the surface charge density on a conductor, $\sigma = \vec n\cdot\vec E/(4\pi)$ (see (1.33)), we have

$$\sigma = -\frac{1}{4\pi}\,\frac{\partial\phi}{\partial r}\Big|_{r=a} = -\frac{q\,(r_1^2 - a^2)}{4\pi a(a^2 + r_1^2 - 2a\,r_1\cos\gamma)^{3/2}}\,. \tag{2.18}$$

If (2.18) is integrated over the area of the sphere, it gives a total charge $q' = -qa/r_1$. In other words, the total induced charge on the surface of the sphere is equal to the *image* charge. This is in accordance with Gauss's law.

Some simple generalisations:

Because of the linearity of the Maxwell equations, it is straightforward to generalise the above result in a variety of ways. For example, instead of taking the conducting sphere to be at zero potential, we could consider a situation where it is held at a non-zero potential V (relative to zero at infinity, still). All that need be done is to add another term to the potential (2.16), corresponding to the introduction of a point charge at the origin. Thus if we now take

$$\phi(\vec{r}) = \frac{q}{|\vec{r}-\vec{r}_1|} - \frac{qa/r_1}{|\vec{r}-(a^2/r_1^2)\,\vec{r}_1|} + \frac{Q}{r}\,, \tag{2.19}$$

then the potential on the surface of the sphere becomes $\phi = Q/a$. Choosing $Q = Va$ therefore gives the required result.

As another generalisation, we can calculate the solution for a grounded sphere placed in a previously-uniform electric field. Without loss of generality, let us take the electric field to be directed along the z axis. The uniform field can be achieved via a limiting process in which two point charges $\pm Q$ are placed at $z = \mp b$ respectively. Close to the origin, there will therefore be an approximately uniform electric field $E_0 \approx 2Q/b^2$ directed along z. Eventually, we take b to infinity, whilst holding $E_0 = 2Q/b^2$ fixed, and the approximation becomes exact.

In the presence of the grounded sphere, each of the above charges will have its image charge, with $+Q$ at $-b$ having an image charge $-Qa/b$ at $z = a^2/b$, and $-Q$ at $+b$ having an image charge $+Qa/b$ at $z = -a^2/b$. If we use spherical polar coordinates to write

$$\vec{r} = (x,y,z) = (r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta)\,, \tag{2.20}$$

then from (2.17) we deduce that the total potential for the system we are considering will be

$$\begin{aligned}\phi(r,\theta,\varphi) &= \frac{Q}{(r^2+b^2+2rb\cos\theta)^{1/2}} - \frac{Q}{(r^2+b^2-2rb\cos\theta)^{1/2}} \qquad (2.21)\\ &\quad -\frac{Qa/b}{(r^2+a^4/b^2+2a^2r/b\,\cos\theta)^{1/2}} + \frac{Qa/b}{(r^2+a^4/b^2-2a^2r/b\,\cos\theta)^{1/2}}\,.\end{aligned}$$

Expanding as a power series in $1/b$, we find

$$\phi = -\frac{2Q}{b^2}\,r\cos\theta + \frac{2Q}{b^2}\,\frac{a^3}{r^2}\,\cos\theta + \cdots \tag{2.22}$$

where the higher-order terms involve higher inverse powers of b and therefore they will go to zero when b is sent to infinity holding $E_0 = 2Q/b^2$ fixed. In this limit, we therefore find that

$$\phi = -E_0 \Big(r - \frac{a^3}{r^2} \Big) \cos\theta \,. \tag{2.23}$$

The first term in (2.23) can be written using Cartesian coordinates as $\phi = -E_0\, z$, and so it just describes the purely uniform electric field $\vec{E} = -\vec{\nabla}\phi = (0,0,E_0)$ that would occur in the absence of the grounded sphere. The second term describes an electric dipole contribution to the potential, arising from the two pairs of charges plus images.

2.4 Dirichlet Green function for spherical boundary

We can use the results in section 2.3 to construct the Dirichlet Green function for the boundary-value problem where the potential is specified on the surface of a sphere. We just need to set $q=1$ and $\vec{r}_1 = \vec{r}\,'$ in (2.16), leading to

$$G_D(\vec{r},\vec{r}\,') = \frac{1}{|\vec{r}-\vec{r}\,'|} - \frac{a/r'}{|\vec{r}-(a^2/r'^2)\,\vec{r}\,'|} \,. \tag{2.24}$$

If we introduce γ as the angle between $\vec{r}$ and $\vec{r}\,'$, then (2.24) can be written as

$$G_D(\vec{r},\vec{r}\,') = \frac{1}{\sqrt{r^2+r'^2-2rr'\,\cos\gamma}} - \frac{1}{\sqrt{a^2+r^2\,r'^2/a^2-2rr'\,\cos\gamma}} \,. \tag{2.25}$$

Written in this form, it is manifest that $G_D(\vec{r},\vec{r}\,')$ is symmetric under the exchange of $\vec{r}$ and $\vec{r}\,'$. It is also manifest that $G_D(\vec{r},\vec{r}\,')$ vanishes, as it should, if $\vec{r}$ or $\vec{r}\,'$ lies on the surface of the sphere.

To use this expression in the general boundary-value integral in (1.90), we need to calculate the normal derivative with respect to $\vec{r}\,'$, evaluated on the sphere at $r'=a$. Bearing in mind that the *outward* normal from the volume V (external to the sphere) is directed *inwards* towards the centre of the sphere, we therefore need

$$\frac{\partial G_D(\vec{r},\vec{r}\,')}{\partial n'}\Big|_{r'=a} = -\frac{\partial G_D(\vec{r},\vec{r}\,')}{\partial r'}\Big|_{r'=a} = -\frac{r^2-a^2}{a\,[r^2+a^2-2ar\cos\gamma]^{3/2}} \,. \tag{2.26}$$

Substituting into (1.90) (and taking $\rho=0$ for simplicity), we obtain

$$\phi(\vec{r}) = \frac{a(r^2-a^2)}{4\pi} \int \frac{\phi(a,\theta',\varphi')}{[r^2+a^2-2ar\cos\gamma]^{3/2}}\, d\Omega' \,, \tag{2.27}$$

where in the boundary integral we express the potential ϕ in terms of spherical polar coordinates (r',θ',φ'). The area element on the sphere of radius a is written as $\vec{n}\cdot d\vec{S}' = a^2 d\Omega'$, where $d\Omega' = \sin\theta'\,d\theta' d\varphi'$ is the area element on the unit sphere (i.e. the solid angle element). The expression (2.27) gives the result for the potential everywhere outside a spherical surface of radius a, on which the potential is specified to be $\phi(a,\theta',\varphi')$.

Note that the integration in (2.27) is actually rather complicated, even if $\phi(a,\theta',\varphi')$ itself is a simple function, because of the $\cos\gamma$ appearing in the denominator. Using spherical polar coordinates, the Cartesian components of $\vec r$ and $\vec r\,'$ are

$$\vec r = (r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta)\,, \qquad \vec r\,' = (r'\sin\theta'\cos\varphi', r'\sin\theta'\sin\varphi', r'\cos\theta')\,, \tag{2.28}$$

and so $\cos\gamma$, which is defined by $\vec r\cdot\vec r\,' = rr'\,\cos\gamma$, is given by

$$\cos\gamma = \sin\theta\sin\theta'\cos(\varphi-\varphi') + \cos\theta\cos\theta'\,. \tag{2.29}$$

Consider, as an example, the case where one hemisphere of the boundary surface is held at a constant potential V, while the other hemisphere is held at potential $-V$. Since we are using standard spherical polar coordinates, it is natural to orient things so that the two hemispheres correspond to the parts of the sphere with $z>0$ and $z<0$ respectively. In other words, we have

$$\begin{aligned} \phi(a,\theta,\varphi) &= +V \qquad \text{for} \quad 0\le\theta<\frac{\pi}{2}\,,\\ \phi(a,\theta,\varphi) &= -V \qquad \text{for} \quad \frac{\pi}{2}<\theta\le\pi\,. \end{aligned} \tag{2.30}$$

Equation (2.27) therefore gives

$$\begin{aligned} \phi(\vec r) &= \frac{aV\,(r^2-a^2)}{4\pi}\int_0^{2\pi}d\varphi'\Big[\int_0^{\pi/2}\frac{\sin\theta'}{(r^2+a^2-2ar\cos\gamma)^{3/2}}\,d\theta'\\ &\qquad -\int_{\pi/2}^{\pi}\frac{\sin\theta'}{(r^2+a^2-2ar\cos\gamma)^{3/2}}\,d\theta'\Big]\,, \end{aligned} \tag{2.31}$$

where $\cos\gamma$ is given by (2.29). By making the change of variables $\theta'\to\pi-\theta'$ and $\varphi'\to\varphi'+\pi$ in the second integral, this can be written as

$$\phi(\vec r) = \frac{aV\,(r^2-a^2)}{4\pi}\int_0^{2\pi}d\varphi'\int_0^{\pi/2}\Big[\frac{\sin\theta'}{(r^2+a^2-2ar\cos\gamma)^{3/2}}-\frac{\sin\theta'}{(r^2+a^2+2ar\cos\gamma)^{3/2}}\Big]\,d\theta' \tag{2.32}$$

Unfortunately, the integrations are too complicated to admit a useful explicit closed-form result.[6]

We can easily integrate (2.32) if we ask for the potential ϕ only in the special case where we are on the positive z axis, i.e. for $\theta=0$. It then follows from (2.29) that $\cos\gamma=\cos\theta'$,

[6] This illustrates an important point, that although we may say that the boundary-value problem for the spherical boundary has been "solved" once we obtained an explicit closed-form result for the Green function, it does not necessarily mean that we can present an explicit closed-form expression for the solution.

and then elementary integration of (2.32) gives, for $z > a$,

$$\phi(z) = V\left(1 - \frac{z^2 - a^2}{z\sqrt{z^2 + a^2}}\right). \tag{2.33}$$

In the absence of a closed-form expression for the general off-axis potential, one could resort to making a power-series expansion of the integrand in (2.32) in powers of $\cos\gamma$, and then performing the integrations term by term. This is a somewhat clumsy approach, and so instead we shall postpone further discussion of this example until a little later in the course, when we shall have developed an approach which will allow us to obtain the power series expression for the off-axis potential very easily.

3 Separation of Variables in Cartesian Coordinates

3.1 Introduction

The boundary-value problem in electrostatics is formulated as the problem of solving Poisson's equation $\nabla^2\phi = -4\phi\rho$ in a volume V bounded by a surface S on which appropriate boundary conditions are imposed. Quite commonly, we are interested in the situation where $\rho = 0$ in V, so that the potential ϕ in V is governed entirely by the conditions that it, or its normal derivative, satisfy on S.

The geometry of the boundary surface S typically dictates what type of coordinate system is best adapted to the problem. For example, if S is formed by one or more planar surfaces, then Cartesian coordinates are likely to be the most convenient choice. If, on the other hand, the boundary S is spherical, then spherical polar coordinates will probably be the best choice. For a boundary of cylindrical shape, cylindrical polar coordinates will be most convenient.

All three of these coordinate systems share the special property that when using them the Laplacian operator ∇^2 is *separable*. This would not be true for some arbitrary choice of coordinate system. The defining property of a separable coordinate system is that Laplace's equation, which is itself a second-order partial differential equation, can be factored into a system of second-order *ordinary* differential equations. This is of enormous benefit when one tries to construct solutions.

We shall describe the process of separation of variables in the three cases of Cartesian, spherical polar, and cylindrical polar, coordinates. In each case, the solution of the factored ordinary differential equations requires an understanding of certain classes of special functions. In the Cartesian case, the relevant special functions are just the familiar sine and cosine trigonometric functions. In the case of spherical polar coordinates, the Legendre and associated Legendre functions arise, whilst in the case of cylindrical polar coordinates it is Bessel functions that arise.

We begin in this section with the separation of variables in Cartesian coordinates.

3.2 Separation of variables in Cartesian coordinates

The Laplace equation in Cartesian coordinates is simply

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0\,. \tag{3.1}$$

The separation of variables is achieved by considering a solution ϕ of the factorised form

$$\phi(x,y,z) = X(x)\,Y(y)\,Z(z)\,. \tag{3.2}$$

Substituting this into (3.1), and dividing out by ϕ, yields

$$\frac{1}{X(x)}\,\frac{d^2X(x)}{dx^2} + \frac{1}{Y(y)}\,\frac{d^2Y(y)}{dy^2} + \frac{1}{Z(z)}\,\frac{d^2Z(z)}{dz^2} = 0\,. \tag{3.3}$$

The first term is independent of y and z, the second is independent of x and z, and the third is independent of x and y. It therefore follows that each term must separately be a constant, with the three constants summing to zero. Therefore either two of the constants are positive with the third negative, or two are negative with the third positive. Let us take the constants in the first two terms to be negative, and the last to be positive, so that we may write

$$\frac{d^2X}{dx^2} + \alpha^2 X = 0\,,\qquad \frac{d^2Y}{dy^2} + \beta^2 Y = 0\,,\qquad \frac{d^2Z}{dz^2} - \gamma^2 Z = 0\,, \tag{3.4}$$

with

$$\gamma^2 = \alpha^2 + \beta^2\,. \tag{3.5}$$

The solutions for X, Y and Z will therefore be of the forms

$$X \sim e^{\pm \mathrm{i}\alpha x}\,,\qquad Y \sim e^{\pm \mathrm{i}\beta y}\,,\qquad Z \sim e^{\pm\gamma z}\,. \tag{3.6}$$

Equivalently, the solutions for X and Y can be taken to be linear combinations of sine and cosine functions of their respective arguments, while Z can be written in terms of hyperbolic functions.

The general solution to (3.1) can now be written as a sum over all the "basic" solutions of the form (3.2) that we have now constructed. Since α and β are at this stage arbitrary constants, the general solution will have the form

$$\phi(x,y,z) = \int_{-\infty}^{\infty} d\alpha \int_{-\infty}^{\infty} d\beta e^{\mathrm{i}(\alpha x+\beta y)} \Big(a(\alpha,\beta)\,e^{\gamma x} + b(\alpha,\beta)\,e^{-\gamma z}\Big)\,, \tag{3.7}$$

where $a(\alpha,\beta)$ and $b(\alpha,\beta)$ are arbitrary functions, and γ is given by (3.5).

The general solution to (3.1) is expressed as an integral over a continuum of the basic solutions, as in (3.7). However, as soon as we also impose boundary conditions on the solution, the continuous integrals will be replaced by a discrete sum over basic solutions.

Example: A rectangular hollow box:

Suppose, for example, we wish to solve Laplace's equation inside a hollow rectangular box, with sides of length a, b and c in the x, y and z directions respectively. We may set up the axes so that the origin is at one corner of the box, so that the faces are located at $x=0$ and $x=a$; at $y=0$ and $y=b$; and at $z=0$ and $z=c$. Suppose that the faces are all held at zero potential, except for the face at $z=c$, on which the potential is specified to be

$$\phi(x,y,c) = V(x,y)\,, \tag{3.8}$$

for some specified voltage profile function $V(x,y)$.

Since the potential vanishes at $x=0$ for all y and z, it follows that we must arrange for $X(x)$ to vanish at $x=0$. Since the general solution for $X(x)$ is

$$X(x) = a_1 e^{\mathrm{i}\alpha x} + b_1 e^{-\mathrm{i}\alpha x}\,, \tag{3.9}$$

this means we must have $b_1 = -a_1$, and so $X(x) \sim \sin\alpha x$. The potential also vanishes at $x=a$ for all y and z, and this means that we must have $X(a)=0$. This implies that α must be restricted to take only a discrete (but infinite) set of values,

$$\alpha = \frac{m\pi}{a}\,, \tag{3.10}$$

where m is any integer. Without loss of generality we may assume that m is a *positive* integer, since the negative values will just reproduce the same set of functions (multiplied by -1).

In the same way, the vanishing of ϕ at $y=0$ and $y=b$ implies that $Y(y)$ must be proportional to $\sin\beta y$, and that β must be of the form

$$\beta = \frac{n\pi}{b}\,, \tag{3.11}$$

where n is any positive integer.

The vanishing of ϕ at $z=0$ implies that $Z(z)$ must be proportional to $\sinh\gamma z$. Since γ is given in terms of α and β by (3.5), it follows that the general solution for ϕ that satisfies all the boundary conditions except the one on the remaining face at $z = c$ can be written as

$$\phi(x,y,z) = \sum_{m\ge1}\sum_{n\ge1} A_{mn}\, \sin\frac{m\pi x}{a}\, \sin\frac{n\pi y}{b}\, \sinh\left(\pi z\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}\right), \tag{3.12}$$

where A_{mn} are arbitrary constants.

The constants A_{mn} are determined by matching ϕ to the given boundary condition (3.8) at $z = c$. This amounts to constructing a two-dimensional Fourier series expansion for the function $V(x, y)$. To do this, it is useful to recall a few facts about expansions in terms of complete sets of orthogonal functions.

3.3 Generalised Fourier expansions

Suppose that the functions $u_n(x)$ for $n \ge 1$ form a *complete set* in the interval $a \le x \le b$. They may be either real, or complex, and so for generality we shall assume that they are complex, with complex conjugation denoted by a bar. We furthermore assume that the functions are *orthogonal*, meaning that

$$\int_a^b \bar u_m(x) u_n(x) dx = 0\,, \qquad m \ne n\,. \tag{3.13}$$

The integral when $m = n$ is non-zero and finite, and so we may for convenience scale the functions so that when $m = n$ the integral gives unity. Thus we may assume *orthonormal* functions, satisfying

$$\int_a^b \bar u_m(x) u_n(x) dx = \delta_{mn}\,. \tag{3.14}$$

Suppose now we have an arbitrary function $f(x)$, which is assumed to be square-integrable on the interval $a \le x \le b$. (i.e. $\int_a^b |f(x)|^2 dx$ is finite.) We may expand $f(x)$ in terms of the complete set of functions $u_n(x)$, by writing

$$\boxed{f(x) = \sum_{n\ge1} a_n u_n(x)\,,} \tag{3.15}$$

where the a_n are certain constants. These constants can be determined by multiplying (3.15) by $\bar u_m(x)$, integrating over the interval $a \le x \le b$, and using the orthonormality relations (3.14):

$$\int_a^b \bar u_m(x) f(x) dx = \sum_{n\ge1} a_n \int_a^b \bar u_m(x) u_n(x) dx = \sum_{n\ge1} a_n \delta_{mn} = a_m\,, \tag{3.16}$$

Thus we have

$$\boxed{a_n = \int_a^b \bar u_n(x) f(x) dx\,.} \tag{3.17}$$

If we plug (3.17) into (3.15), being careful to distinguish between the argument x in (3.15) and the integration variable x in (3.17) (which we shall now call x'), we obtain

$$f(x) = \int_a^b \Big(\sum_{n\ge 1} \bar u_n(x') u_n(x)\Big)\, f(x') dx'\,. \tag{3.18}$$

Comparing this with the defining property of the delta function, given in (1.50), we see that we may conclude that

$$\boxed{\sum_{n\ge 1} \bar u_n(x') u_n(x) = \delta(x'-x)\,.} \tag{3.19}$$

The delta function is an *even* function, and so this is also equal to $\delta(x-x')$.

The generalisation to expansions in more than one dimension is immediate. For example, suppose in two dimensions we have the orthonormal functions $u_m(x)$ as before, defined in $a\le x\le b$, and another orthonormal set $v_n(y)$ defined in the interval $c\le y\le d$. We can expand

$$f(x,y) = \sum_{m\ge 1}\sum_{n\ge 1} a_{mn}\, u_m(x) v_n(y)\,, \tag{3.20}$$

and read off the coefficients a_{mn} by first multiplying by $\bar u_p(x)\bar v_q(y)$ and integrating, to give

$$a_{mn} = \int_a^b dx \int_c^d dy\, \bar u_m(x) \bar v_n(y) f(x,y)\,. \tag{3.21}$$

Now we can go back to our boundary-value problem in the rectangular box. Recall that we had obtained the expression (3.12) for $\phi(x,y,z)$ everywhere inside the box, expressed as a double summation. It remained for us to determine the expansion coefficients A_{mn}, by matching $\phi(x,y,z)$ to the given boundary potential $V(x,y)$ at $z=c$. In other words, we must find A_{mn} such that

$$V(x,y) = \sum_{m\ge 1}\sum_{n\ge 1} A_{mn}\, \sin\frac{m\pi x}{a}\, \sin\frac{n\pi y}{b}\, \sinh\Big(\pi c \sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}\Big)\,, \tag{3.22}$$

This amounts to solving for the coefficients a_{mn} such that

$$V(x,y) = \sum_{m\ge 1}\sum_{n\ge 1} a_{mn} \sin\frac{m\pi x}{a}\, \sin\frac{n\pi y}{b}\,, \tag{3.23}$$

since then the A_{mn} will be given by

$$A_{mn} = \frac{a_{mn}}{\sinh\Big(\pi c\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}\Big)}\,. \tag{3.24}$$

To determine the coefficients a_{mn} in (3.23), we recognise that this is an example of a two-dimensional expansion in terms of the complete sets of functions $\sin\frac{m\pi x}{a}$ and $\sin\frac{n\pi y}{b}$, and so to read off a_{mn} we just need to multiply by $\sin\frac{p\pi x}{a}\sin\frac{q\pi y}{b}$ and integrate. The functions $\sin\frac{m\pi x}{a}$ are not normalised. We can either scale them appropriately to normalise them, or else work with unnormalised functions and take care to keep track of the additional factors that this will give. The latter is the simpler way to proceed. We just need to note that

$$\begin{aligned}\int_0^a \sin\frac{m\pi x}{a}\sin\frac{p\pi x}{a}dx &= \tfrac{1}{2}\int_0^a \Big(\cos\frac{(m-p)\pi x}{a} - (\cos\frac{(m+p)\pi x}{a}\Big)dx\,, \qquad (3.25)\\ &= \Big[\frac{a}{2(m-p)\pi}\sin\frac{(m-p)\pi x}{a} - \frac{a}{2(m+p)\pi}\sin\frac{(m+p)\pi x}{a}\Big]_0^a = 0\end{aligned}$$

when $m\neq p$, whilst

$$\int_0^a \sin^2\frac{m\pi x}{a}dx = \int_0^a \Big(\tfrac{1}{2}-\tfrac{1}{2}\cos\frac{2m\pi x}{a}\Big)dx = \tfrac{1}{2}a - \frac{a}{4m\pi}\Big[\sin\frac{2m\pi x}{a}\Big]_0^a = \tfrac{1}{2}a\,. \qquad (3.26)$$

Thus we have

$$\boxed{\int_0^a \sin\frac{m\pi x}{a}\sin\frac{p\pi x}{a}dx = \tfrac{1}{2}a\,\delta_{mp}\,. \qquad (3.27)}$$

Using this, we deduce from (3.23) that

$$a_{mn} = \frac{4}{ab}\int_0^a dx\int_0^b dy\, V(x,y)\sin\frac{m\pi x}{a}\,\sin\frac{n\pi y}{b}\,. \qquad (3.28)$$

4 Separation of variables in spherical polar coordinates

The spherical polar coorinates (r,θ,φ) are related to Cartesian coordinates (x,y,z) by

$$x = r\,\sin\theta\,\cos\phi\,,\qquad y = r\,\sin\theta\,\sin\phi\,,\qquad z = r\,\cos\theta\,. \qquad (4.1)$$

In terms of (r,θ,φ), Laplace's equation (3.1) becomes[7]

$$\frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2\frac{\partial\phi}{\partial r}\Big) + \frac{1}{r^2}\nabla^2_{(\theta,\varphi)}\phi = 0\,, \qquad (4.2)$$

where $\nabla^2_{(\theta,\varphi)}$ is the two-dimensional Laplace operator on the surface of the unit-radius sphere,

$$\nabla^2_{(\theta,\varphi)} \equiv \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\Big(\sin\theta\frac{\partial}{\partial\theta}\Big) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\,. \qquad (4.3)$$

Laplace's equation can be separated by first writing $\phi(r,\theta,\varphi)$ in the form

$$\phi(r,\theta,\varphi) = \frac{1}{r}R(r)\,Y(\theta,\varphi)\,. \qquad (4.4)$$

[7] This can be seen by using the chain rule to convert from derivatives with respect to x, y and z to derivatives with respect to r, θ and φ. Thus, $\partial/\partial x = (\partial r/\partial x)\partial/\partial r + (\partial\theta/\partial x)\partial/\partial\theta + (\partial\varphi/\partial x)\partial/\partial\varphi$, etc.

Substituting into (4.2), and dividing out by ϕ, we get

$$\frac{r^2}{R}\frac{d^2R}{dr^2} + \frac{1}{Y}\nabla^2_{(\theta,\varphi)}Y = 0\,. \tag{4.5}$$

(It is useful to note that $r^{-2}\partial(r^2\partial\phi/\partial r)/\partial r$ can be rewritten as $r^{-1}\partial^2(r\,\phi)/\partial r^2$, when doing this calculation.)

The last term in (4.5) depends only on θ and φ, while the first term depends only on r, and so consistency for all (r,θ,φ) therefore means that each term must be constant, and so

$$\nabla^2_{(\theta,\varphi)}\,Y = -\lambda\,Y\,,\qquad \frac{d^2R}{dr^2} = \frac{\lambda}{r^2}\,R\,. \tag{4.6}$$

The key point now is that one can show that the harmonics $Y(\theta,\varphi)$ on the sphere are well-behaved only if the separation constant λ takes a certain discrete infinity of non-negative values. The most elegant way to show this is by making use of the symmetry properties of the sphere, but since this takes us away from the main goals of the course, we shall not follow that approach here.[8] Instead, we shall follow the more "traditional," if more pedestrian, approach of examining the conditions under which singular behaviour of the eigenfunction solutions of the differential equation can be avoided.

To study the eigenvalue problem $\nabla^2_{(\theta,\phi)}\,Y = -\lambda\,Y$ in detail, we make a further separation of variables by taking $Y(\theta,\varphi)$ to be of the form $Y(\theta,\varphi) = \Theta(\theta)\,\Phi(\varphi)$. Substituting this in, and multiplying by $\sin^2\theta\,Y^{-1}$, we get

$$\frac{1}{\Theta}\,\sin\theta\,\frac{d}{d\theta}\Big(\sin\theta\,\frac{d\Theta}{d\theta}\Big) + \lambda\,\sin^2\theta + \frac{1}{\Phi}\,\frac{d^2\Phi}{d\varphi^2} = 0\,. \tag{4.7}$$

By now-familiar arguments the last term depends only on φ, while the first two depend only on θ. Consistency for all θ and φ therefore implies that the last term must be a constant,

[8] The essential point is that the surface of the unit sphere can be defined as $x^2+y^2+z^2=1$, and this is invariant under transformations of the form

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longrightarrow M \begin{pmatrix} x \\ y \\ z \end{pmatrix}\,,$$

where M is any constant 3×3 orthogonal matrix, satisfying $M^T\,M = \mathbb{1}$. This shows that the sphere is invariant under the 3-parameter group $O(3)$, and hence the eigenfunctions Y must fall into representations under $O(3)$. The calculation of the allowed values for λ, and the forms of the associated eigenfunctions Y, then follow from group-theoretic considerations. Anticipating the result that we shall see by other means, the eigenvalues λ take the form $\lambda_\ell = \ell(\ell+1)$, where ℓ is any non-negative integer. The eigenfunctions are classified by ℓ and a second integer m, with $-\ell \le m \le \ell$, and are the well-known spherical harmonics $Y_{\ell m}(\theta,\phi)$. The fact that λ depends on ℓ but not m means that the eigenvalue $\lambda_\ell = \ell(\ell+1)$ has a degeneracy $(2\ell+1)$.

and so we have

$$\begin{aligned}\frac{d^2\Phi}{d\varphi^2} + m^2\,\Phi &= 0\,, &(4.8)\\ \sin\theta\,\frac{d}{d\theta}\Big(\sin\theta\,\frac{d\Theta}{d\theta}\Big) + (\lambda\,\sin^2\theta - m^2)\,\Theta &= 0\,. &(4.9)\end{aligned}$$

The solution to the Φ equation is $\Phi \sim e^{\pm \mathrm{i}\, m\,\varphi}$. The constant m^2 could, *a priori*, be positive or negative, but we must recall that the coordinate φ is periodic on the sphere, with period 2π. The periodicity implies that the eigenfunctions Φ should be periodic too, and hence it must be that m^2 is non-negative. In order that we have $\Phi(\varphi + 2\pi) = \Phi(\varphi)$ it must furthermore be the case that m is an integer.

4.1 Series solution of the Legendre equation

To analyse the eigenvalue equation (4.9) for Θ, it is advantageous to define a new independent variable x, related to θ by $x = \cos\theta$. (Do not confuse this variable x with the Cartesian coordinate x!) At the same time, let us now use y instead of Θ as our symbol for the dependent variable. Equation (4.9) therefore becomes

$$\boxed{\frac{d}{dx}\Big((1-x^2)\,\frac{dy}{dx}\Big) + \Big(\lambda - \frac{m^2}{1-x^2}\Big)\,y = 0\,. \qquad (4.10)}$$

This equation is called the *Associated Legendre Equation*, and it will become necessary to study its properties, and solutions, in some detail in order to be able to construct solutions of Laplace's equation in spherical polar coordinates. In fact, as we shall see, it is convenient first to study the simpler equation when $m = 0$, which corresponds to the case where the harmonics $Y(\theta, \varphi)$ on the sphere are independent of the azimuthal angle φ. The equation (4.10) in the case $m = 0$ is called the *Legendre Equation.*

Taking $m = 0$ for now, the associated Legendre equation (4.10) reduces to the Legendre equation

$$[(1-x^2)\,y']' + \lambda\,y = 0\,.\,, \qquad (4.11)$$

which we can also write as

$$\boxed{(1-x^2)\,y'' - 2x\,y' + \lambda\,y = 0\,. \qquad (4.12)}$$

Note that here we are denoting a derivative with respect to x by a prime, so that dy/dx is written as y', and so on. We can construct the solutions to (4.12) by applying a procedure known as the *Frobenius Method*, in which $y(x)$ is obtained as a power series in x.

The general theory of how to construct series solutions to ordinary differential equations is quite involved. However, in the present case things are very simple, because the point $x = 0$ around which we wish to expand the power series is a so-called *ordinary point* of the differential equation. (i.e. if the equation is written in the form $y'' + P(x)y' + Q(x)y = 0$ by dividing out by $(1 - x^2)$, the functions $P = -2x/(1 - x^2)$ and $Q = \lambda/(1 - x^2)$ are regular and analytic around $x = 0$.) This means that the two independent solutions to (4.12) can both be expanded in Taylor series around $x = 0$.[9]

Thus, we begin by writing the series expansion

$$y = \sum_{n\geq 0} a_n\, x^n\,. \tag{4.13}$$

Clearly we shall have

$$y' = \sum_{n\geq 0} n\, a_n\, x^{n-1}\,, \qquad y'' = \sum_{n\geq 0} n\,(n-1)\, a_n\, x^{n-2}\,. \tag{4.14}$$

Substituting into equation (4.12), we find

$$\sum_{n\geq 0} n\,(n-1)\, a_n\, x^{n-2} + \sum_{n\geq 0} (\lambda - n\,(n+1))\, a_n\, x^n = 0\,. \tag{4.15}$$

Since we want to equate terms order by order in x, it is useful to shift the summation variable by 2 in the first term, by writing $n = m + 2$;

$$\sum_{n\geq 0} n\,(n-1)\, a_n\, x^{n-2} = \sum_{m\geq -2} (m+2)(m+1)\, a_{m+2}\, x^m = \sum_{m\geq 0} (m+2)(m+1)\, a_{m+2}\, x^m\,. \tag{4.16}$$

(The last step, where we have dropped the $m = -2$ and $m = -1$ terms in the summation, clearly follows from the fact that the $(m + 2)(m + 1)$ factor gives zero for these two values of m.) Finally, relabelling m as n again, we get from (4.15)

$$\sum_{n\geq 0} \Big((n+2)(n+1)\, a_{n+2} + (\lambda - n\,(n+1))\, a_n\Big)\, x^n = 0\,. \tag{4.17}$$

[9] In more general cases where one is expanding around a singular point of the equation (say at $x = b$), one needs to consider a series of the form $y(x) = \sum_{n\geq 0} a_n (x - b)^{n+\sigma}$, where σ is a constant that may be non-integer. It is rather common to see people automatically considering this more general type of expansion, even when they are expanding around an ordinary point of the equation. Although this is not actually wrong, it is rather inconvenient, since it represents an over-parameterisation of the problem and therefore it obscures the essential simplicity of the procedure. Possibly they do it because they are unaware of the theorem that both solutions of a second-order ordinary differential equation are analytic in the neighbourhood of an ordinary point.

Since this must hold for all values of x, it follows that the coefficient of each power of x must vanish separately, giving

$$(n+2)(n+1)\,a_{n+2} + (\lambda - n\,(n+1))\,a_n = 0 \qquad (4.18)$$

for all $n \ge 0$. Thus we have the recursion relation

$$\boxed{a_{n+2} = \frac{n\,(n+1) - \lambda}{(n+1)(n+2)}\,a_n\,. \qquad (4.19)}$$

We see from (4.19) that all the coefficients a_n with $n \ge 2$ can be solved for, in terms of a_0 and a_1. In fact all the a_n for *even* n can be solved for in terms of a_0, while all the a_n for *odd* n can be solved for in terms of a_1. Since the equation is linear, we can take the even-n series and the odd-n series as the two linearly independent solutions of the Legendre equation, which we can call $y_{(1)}(x)$ and $y_{(2)}(x)$:

$$\begin{aligned} y_{(1)}(x) &= a_0 + a_2\,x^2 + a_4\,x^4 + \cdots\,, \\ y_{(2)}(x) &= a_1 + a_3\,x^3 + a_5\,x^5 + \cdots\,. \end{aligned} \qquad (4.20)$$

The first solution involves only the even a_n, and thus has only even powers of x, whilst the second involves only the odd a_n, and has only odd powers of x. We can conveniently consider the two solutions separately, by taking either $a_1 = 0$, to discuss $y_{(1)}$, or else taking $a_0 = 0$, to discuss $y_{(2)}$.

Starting with $y_{(1)}$, we therefore have from (4.19) that $a_2 = -\frac{1}{2}\lambda\,a_0$, $a_3 = 0$, $a_4 = \frac{1}{12}(6-\lambda)\,a_2$, $a_5 = 0$, etc. In the expression for a_4, we can substitute the expression already found for a_2, and so on. Thus we shall get

$$\begin{aligned} a_2 &= -\tfrac{1}{2}\lambda\,a_0\,, \qquad a_4 = -\tfrac{1}{12}\lambda\,(6-\lambda)\,a_0\,, \qquad \ldots \\ a_3 &= a_5 = a_7 = \cdots = 0\,. \end{aligned} \qquad (4.21)$$

The series solution in this case is therefore given by

$$y_{(1)} = a_0\,\Big(1 - \tfrac{1}{2}\lambda\,x^2 - \tfrac{1}{24}\lambda\,(6-\lambda)\,x^4 + \cdots\Big)\,. \qquad (4.22)$$

To discuss the solution $y_{(2)}$ instead, we can take $a_0 = 0$ and $a_1 \ne 0$. The recursion relation (4.19) now gives $a_2 = 0$, $a_3 = \frac{1}{6}(2-\lambda)\,a_1$, $a_4 = 0$, $a_5 = \frac{1}{20}(12-\lambda)\,a_3$, $a_5 = 0$, etc, and so we find

$$\begin{aligned} a_3 &= \tfrac{1}{6}(2-\lambda)\,a_1\,, \qquad a_5 = \tfrac{1}{120}(2-\lambda)\,(12-\lambda)\,a_1\,, \qquad \ldots \\ a_2 &= a_4 = a_6 = \cdots = 0\,. \end{aligned} \qquad (4.23)$$

The series solution in this case therefore has the form

$$y_{(2)} = a_1 \Big(x + \tfrac{1}{6}(2-\lambda)\, x^3 + \tfrac{1}{120}(2-\lambda)\,(12-\lambda)\, x^5 + \cdots \Big) . \tag{4.24}$$

To summarise, we have produced two linearly independent solutions to our differential equation (4.12), which are given by (4.22) and (4.24). The fact that they are independent is obvious, since the first is an even function of x whilst the second is an odd function.

So far in the discussion, the separation constant λ has been allowed to be completely arbitrary. As we shall now see, it must be restricted to take a discrete infinite set of values in order to have solutions that are non-singular. In other words, λ can only take values in this discrete set if one insists that the power series expansion should converge for all relevant values of x.

The convergence of an infinite series can be tested by applying the *ratio test*. The statement of this test is that the series converges if the ratio of successive terms in the series is of magnitude less than 1, in the limit as one looks further and further down the series. If, on the other hand, the ratio is greater than 1, then the series diverges. If the ratio equals 1, then the test is inconclusive.

Since the two independent series we obtained correspond to the even powers of x and the odd power of x in (4.13), the ratio of successive terms in either series will be

$$\begin{aligned} R_n &= \frac{a_{n+2} x^{n+2}}{a_n x^n} = \frac{a_{n+2} x^2}{a_n} , \\ &= \frac{n(n+1)-\lambda}{(n+1)(n+2)}\, x^2 . \end{aligned} \tag{4.25}$$

(We used the recursion relation (4.19) in getting to the second line.) In the limit when $n \to \infty$, holding λ fixed, we therefore have

$$R_\infty = x^2 . \tag{4.26}$$

Thus the series (4.13) converges if $|x| < 1$ and diverges if $|x| > 1$. The test is inconclusive for $x = \pm 1$.

Recalling that $x = \cos\theta$ here, we see that the values $x = \pm 1$ are in fact attained in the physical problems we are studying, since these values correspond to $\theta = 0$ and $\theta = \pi$ (the north and south poles of the sphere). It is therefore important to establish how the series expansion behaves at $x = \pm 1$.

We shall not present a detailed discussion of the convergence here, but we simply state the result, which is that in fact, for generic values of λ, the series (4.13) diverges at $x = \pm 1$; i.e. both solutions (even and odd) diverge. Thus, in order to obtain regular solutions for the Legendre equation, we must instead arrange, by judicious choice of the values for λ, to make the series terminate.

Looking at the recursion relation (4.19), it is evident that the series will terminate if λ is chosen to be given by

$$\lambda = \ell(\ell+1)\,, \qquad \ell = \text{integer} \ge 0\,. \tag{4.27}$$

Then, we shall have that the coefficients a_n are non-zero for $n < \ell$, but a_n will vanish for $n = \ell$, All the a_n will be zero for $n > \ell$. Thus, we obtain a polynomial solution, of degree ℓ, when λ satisfies (4.27). Obviously, since it is a sum of a finite numer of terms, the polynomial solution is non-singular for all x.

Note that if ℓ is an even integer then it is the *even* series that terminates to give a finite polynomial solution, but the odd series does not terminate. Conversely, if ℓ is an odd integer then it is the odd series that terminates, while the even series does not. Thus we only ever get one terminating polynomial solution, for each integer ℓ.

As an example to illustrate the divergent behaviour if the series does not terminate, consider the odd series $y_2(x)$, with $\ell = 0$, i.e. $\lambda = 0$. From (4.19) we then have $a_{n+2} = n\,a_n/(n+2)$ (with n odd), which has the solution $a_n = a_1/n$. Thus the series (4.13) becomes

$$y = a_1\,(x + \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 + \tfrac{1}{7}x^7 + \cdots)\,, \tag{4.28}$$

which can be recognised as the power-series expansion of

$$y = \tfrac{1}{2}a_1\,\log\Big(\frac{1+x}{1-x}\Big)\,, \tag{4.29}$$

which clearly diverges at $x = \pm 1$. For all other values of λ that lead to non-terminating series, one similarly finds a logarithmic divergence at $x = \pm 1$.

To summarise, we have established that if λ is given by (4.27), there exists one polynomial solution to the Legendre equation, and it is in particular regular for all $-1 \le x \le 1$. This solution is called the *Legendre Polynomial* $P_\ell(x)$, satisfying

$$(1-x^2)\,\frac{d^2P_\ell(x)}{dx^2} - 2x\,\frac{dP_\ell(x)}{dx} + \ell(\ell+1)\,P_\ell(x) = 0\,. \tag{4.30}$$

By convention, the Legendre polynomial $P_\ell(x)$ is normalised so that

$$P_\ell(1) = 1\,. \tag{4.31}$$

The first few are therefore given by

$$\begin{aligned} &P_0(x) = 1\,, \qquad P_1(x) = x\,, \qquad P_2(x) = \tfrac{1}{2}(3x^2-1)\,, \\ &P_3(x) = \tfrac{1}{2}(5x^3-3x)\,, \qquad P_4(x) = \tfrac{1}{8}(35x^4 - 30x^2 + 3)\,. \end{aligned} \tag{4.32}$$

With λ given by (4.27), and with the corresponding Legendre polynomial normalised according to (4.31), it is not hard to solve the recursion relation (4.19) explicitly, giving the result that

$$\boxed{P_\ell(x) = \sum_{k=0}^{[\ell/2]} \frac{(-1)^k\,(2\ell-2k)!}{2^\ell\,k!\,(\ell-k)!\,(\ell-2k)!}\,x^{\ell-2k}\,,} \qquad (4.33)$$

where $[\ell/2]$ is the integer part of $\ell/2$.[10]

The expression (4.33) for the ℓ'th Legendre polynomial is somewhat unwieldy, and it is often useful to have alternative ways of writing $P_\ell(x)$. We shall give two such alternative expressions in the next two subsections.

4.2 Rodrigues formula

First, noting that $d^\ell x^p/dx^\ell = p!\,x^{p-\ell}/(p-\ell)!$, we observe that (4.33) can be written as

$$P_\ell(x) = \frac{d^\ell}{dx^\ell} \sum_{k=0}^{[\ell/2]} \frac{(-1)^k}{2^\ell\,k!\,(\ell-k)!}\,x^{2\ell-2k}\,, \qquad (4.34)$$

and that the summation can then be extended up to $k=\ell$ since the extra terms are all zero. (The d^ℓ/dx^ℓ kills off all the extra terms in the sum, since they all have fewer than ℓ powers of x.) Inserting and dividing out by a factor of $\ell!$ thus gives

$$P_\ell(x) = \frac{1}{2^\ell\,\ell!}\,\frac{d^\ell}{dx^\ell} \sum_{k=0}^{\ell} \frac{(-1)^k\,\ell!}{k!\,(\ell-k)!}\,x^{2\ell-2k}\,, \qquad (4.35)$$

and we can recognise the sum as the binomial expansion of $(x^2-1)^\ell$. Thus we arrive at *Rodrigues' formula* for $P_\ell(x)$:

$$\boxed{P_\ell(x) = \frac{1}{2^\ell\,\ell!}\,\frac{d^\ell}{dx^\ell}\,(x^2-1)^\ell\,.} \qquad (4.36)$$

Having shown that the expessions (4.33) and (4.36) agree, we can now easily confirm that $P_\ell(x)$ so defined does indeed satisfy the normalisation (4.31). To do this, write (4.36) as

$$P_\ell(x) = \frac{1}{2^\ell\,\ell!}\,\frac{d^\ell}{dx^\ell}\Big[(x-1)^\ell\,(x+1)^\ell\Big]\,, \qquad (4.37)$$

[10] To see this, read off the coefficients a_n in (4.13) by equating the coefficients of each power of x with those in (4.33), and then show that these expressions for a_n indeed satisfy the recursion relation (4.19). This proves that (4.33) indeed satisfies the Legendre equation (4.30). The only slightly tricky point is establishing that the normalisation in (4.33) is indeed such that $P_\ell(x)$ satisfies (4.31). We give a simple proof of this in the next section.

and note that when we differentiate the product $(x-1)^\ell\,(x+1)^\ell$ a total of ℓ times, the *only* term that survives after then setting $x=1$ is the term where all ℓ derivatives land on $(x-1)^\ell$. Since $d^\ell(x-1)^\ell/dx^\ell=\ell!$, we see that indeed

$$P_\ell(1)=\frac{1}{2^\ell\,\ell!}\,\ell!(1+1)^\ell=1\,. \tag{4.38}$$

4.3 The generating function

Another very useful way of representing the Legendre polynomials is by means of a *Generating Function.* The claim is that the Legendre polynomials $P_\ell(x)$ satisfy

$$\boxed{G(x,t)\equiv(1-2x\,t+t^2)^{-1/2}=\sum_{\ell\ge 0}t^\ell\,P_\ell(x)\,,} \tag{4.39}$$

where, for convergence of the series, we must have $|t|<1$. We can see how this is working by looking at the first few terms in the power-series expansion of the left-hand side in powers of t, giving

$$G(x,t)=1+x\,t+\tfrac12(3x^2-1)\,t^2+\tfrac12(5x^3-3x)\,t^3+\tfrac18(35x^4-30x^2+3)\,t^4+\cdots\,. \tag{4.40}$$

Equating this with the right-hand side of (4.39), and comparing the coefficients of each power of t, we read off

$$P_0(x)=1\,,\quad P_1(x)=x\,,\quad P_2(x)=\tfrac12(3x^2-1)\,,\quad P_3(x)=\tfrac12(5x^3-3x) \tag{4.41}$$

and so on, which is precisely what we were finding in (4.32).

To prove that (4.39) correctly generates *all* the Legendre polynomials, we note that if $P_\ell(x)$ satisfies the Legendre equation (4.30) for all ℓ, then multiplying by t^ℓ and summing over ℓ implies that H defined by

$$H\equiv\sum_{\ell\ge0}t^\ell\,[(1-x^2)\,P''_\ell-2x\,P'_\ell+\ell(\ell+1)\,P_\ell] \tag{4.42}$$

should vanish. Now, looking at (4.39) we can see that H can be written as

$$H=(1-x^2)\,\frac{\partial^2G(x,t)}{\partial x^2}-2x\,\frac{\partial G(g,t)}{\partial x}+t\,\frac{\partial^2(t\,G(x,t))}{\partial t^2}\,. \tag{4.43}$$

(The three terms here correlate exactly with the three terms in H.) It is now just a simple exercise in differentiation to show that indeed we have $H=0$, which proves that the functions $P_\ell(x)$ defined by (4.39) satisfy the Legendre equation. They are clearly polynomials, because the power-series expansion of the left-hand side of (4.39) in powers of t will clearly have x-dependent coefficients that are polynomial in x.

Finally, we must check the normalisation, i.e. that $P_\ell(1) = 1$. This is very easy; we just set $x = 1$ in (4.39), to get

$$(1 - 2t + t^2)^{-1/2} = \sum_{\ell \geq 0} t^\ell \, P_\ell(1) \,. \tag{4.44}$$

But the left-hand side is just $(1-t)^{-1}$, which has the binomial expansion

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + t^4 + \cdots = \sum_{\ell \geq 0} t^\ell \,, \tag{4.45}$$

and so by comparing with the right-hand side in (4.44) we immediately get $P_\ell(1) = 1$.

4.4 Expansion in Legendre polynomials

Recall that our goal is to construct solutions of Laplace's equation (4.2) written in spherical polar coordinates, and that we have established that for azimuthally-symmetric solutions the relevant functions in the θ direction are expressed in terms of Legendre polynomials. Constructing a general azimuthally-symmetric solution will therefore require that we expand a general function of θ as a sum over Legendre polynomials $P_\ell(\cos\theta)$. In terms of $x = \cos\theta$, the first task then is to expand a general function $f(x)$ in the form

$$\boxed{f(x) = \sum_{\ell \geq 0} a_\ell \, P_\ell(x) \,. \qquad (4.46)}$$

We can establish the following properties of the Legendre polynomials. Firstly,

$$\int_{-1}^{1} dx P_\ell(x) P_{\ell'}(x) = 0 \,, \qquad \text{if} \quad \ell \neq \ell' \,. \tag{4.47}$$

This can be seen by taking the Legendre equation (4.30) and multiplying it by $P_{\ell'}(x)$, then subtracting the same expression but with ℓ and ℓ' interchanged, and finally, integrating over the interval $-1 \leq x \leq 1$. This gives

$$\begin{aligned} \int_{-1}^{1} dx \Big[P_{\ell'}(x) \frac{d}{dx}\Big((1-x^2)\frac{dP_\ell(x)}{dx}\Big) - P_\ell(x)\frac{d}{dx}\Big((1-x^2)\frac{dP_{\ell'}(x)}{dx}\Big) \\ + [\ell(\ell+1) - \ell'(\ell'+1)] \, P_\ell(x) P_{\ell'}(x) \Big] = 0 \,. \end{aligned} \tag{4.48}$$

Integrating by parts shows that the first two terms cancel, leaving

$$[\ell(\ell+1) - \ell'(\ell'+1)] \int_{-1}^{1} dx \, P_\ell(x) P_{\ell'}(x) = 0 \,. \tag{4.49}$$

Thus if $\ell \neq \ell'$, we see that the orthogonality condition (4.47) must hold.

We also need to know what the integral in (4.47) gives when $\ell = \ell'$. Here is a slightly unorthodox derivation, which is actually quite elegant. Using the generating function (4.39) twice over, once for $\sum_\ell t^\ell P_\ell(x)$ and once for $\sum_{\ell'} s^{\ell'} P_{\ell'}(x)$, we can deduce that

$$\int_{-1}^{1} dx (1 - 2xt + t^2)^{-1/2} (1 - 2xs + s^2)^{-1/2} = \sum_{\ell\ge0} \sum_{\ell'\ge0} t^\ell s^{\ell'} \int_{-1}^{1} dx P_\ell(x) P_{\ell'}(x)\,. \qquad (4.50)$$

Setting $s = t$, and using the already-established orthogonality result (4.47), we find

$$\int_{-1}^{1} \frac{dx}{1 - 2xt + t^2} = \sum_{\ell\ge0} t^{2\ell} \int_{-1}^{1} dx (P_\ell(x))^2\,. \qquad (4.51)$$

Performing the integral on the left-hand side gives

$$\frac{1}{t} \log\Big(\frac{1+t}{1-t}\Big) = \sum_{\ell\ge0} t^{2\ell} \int_{-1}^{1} dx (P_\ell(x))^2\,, \qquad (4.52)$$

and expanding the logarithm in a Taylor series implies

$$\sum_{p\ge0} \frac{2t^{2p}}{2p+1} = \sum_{\ell\ge0} t^{2\ell} \int_{-1}^{1} dx (P_\ell(x))^2\,. \qquad (4.53)$$

Equating the coefficients of each power of t then gives[11]

$$\int_{-1}^{1} dx (P_\ell(x))^2 = \frac{2}{2\ell+1}\,. \qquad (4.55)$$

In summary, we have shown that

$$\boxed{\int_{-1}^{1} dx P_\ell(x) P_{\ell'}(x) = \frac{2}{2\ell+1}\, \delta_{\ell,\ell'}\,. \qquad (4.56)}$$

With this result, we can now determine the coefficients A_ℓ in the generalised Fourier expansion (4.46). Multiplying this equation by $P_n(x)$, integrating over the interval $-1 \le x \le 1$, and using (4.56), we find

$$\int_{-1}^{1} dx f(x) P_n(x) = \sum_{\ell\ge0} a_\ell \int_{-1}^{1} dx P_\ell(x) P_n(x) = \sum_{\ell\ge0} \frac{2}{2\ell+1}\, \delta_{\ell n}\, a_\ell = \frac{2}{2n+1}\, a_n\,, \qquad (4.57)$$

and hence the coefficients in (4.46) are given by

$$\boxed{a_\ell = \tfrac{1}{2}(2\ell+1) \int_{-1}^{1} dx f(x) P_\ell(x)\,. \qquad (4.58)}$$

[11] With a little more work, one can perform the integral on the left-hand side of (4.50) *without* setting $s = t$. This gives

$$\frac{1}{\sqrt{st}} \Big[2\log(1+\sqrt{st}) - \log(1-st)\Big] = \sum_{p\ge0} \frac{2}{2p+1}\, (st)^p\,, \qquad (4.54)$$

and so matching the powers of s and t with those on the right-hand side of (4.50), one can deduce immediately the orthogonality (4.47) *and* the result (4.55).

4.5 Azimuthally-symmetric solutions of Laplace's equation

With these preliminaries, we can now return to the original problem, of finding solutions to Laplace's equation in spherical polar coordinates. For now, we are restricting attention to the case where the problem has azimuthal symmetry, meaning that it is independent of the azimuthal coordinate φ. This means that we take $m = 0$ in (4.7), and so the original potential ϕ has been separated by writing

$$\phi(r,\theta) = \frac{1}{r}\, R(r)\, \Theta(\theta)\,, \tag{4.59}$$

with R and Θ satisfying

$$\frac{d^2R}{dr^2} = \frac{\lambda}{r^2}\, R\,, \qquad \frac{1}{\sin\theta}\, \frac{d}{d\theta}\Big(\sin\theta\, \frac{d\Theta}{d\theta}\Big) + \lambda\, \Theta = 0\,. \tag{4.60}$$

(See equations (4.6) and (4.7).) As we now know, the equation for Θ becomes the Legendre equation (4.10) after making the change of variable $\cos\theta = x$, and its regular solutions are the Legendre polynomials $P_\ell(x)$, occurring when $\lambda = \ell(\ell+1)$.

It is easy to see, by looking for trial solutions of the form $R(r) = r^\alpha$, that the two linearly independent solutions of the radial equation in (4.60) are given by

$$R = r^{\ell+1} \qquad \text{and} \qquad R = r^{-\ell}\,. \tag{4.61}$$

Thus by summing over all possible factorised solutions of the form (4.59), we arrive at the general azimuthally-symmetric solution

$$\boxed{\phi(r,\theta) = \sum_{\ell\ge 0} \Big(A_\ell\, r^\ell + B_\ell\, r^{-\ell-1}\Big)\, P_\ell(\cos\theta)\,,} \tag{4.62}$$

where A_ℓ and B_ℓ are arbitrary constants. These will be determined by the boundary conditions that specify a particular physical configuration.

Exterior Problem:

Suppose, for example, we are considering the *exterior* problem, in which the potential is specified on a spherical surface at $r = a$, and one wishes to calculate the potential at all points outside this surface. It follows therefore that the potential should go to zero at infinity, and so we must have

$$A_\ell = 0\,. \tag{4.63}$$

(Strictly, we could allow A_0 to be non-zero since a non-zero constant potential at infinity would be allowable, but there is no particular point in doing so, since the addititive constant has no physical effect.) The B_ℓ coefficients are determined in the manner we established in the previous subsection. Note that after transforming back to the variable θ from the variable $x = \cos\theta$, the integral (4.56) becomes

$$\boxed{\int_0^\pi d\theta \sin\theta\, P_\ell(\cos\theta) P_{\ell'}(\cos\theta) = \frac{2}{2\ell+1}\,\delta_{\ell,\ell'}\,,} \tag{4.64}$$

and so the coefficients B_ℓ in (4.62) are given in terms of the boundary values $\phi(a,\theta)$ of the potential by

$$B_\ell = \tfrac{1}{2}(2\ell+1)\, a^{\ell+1} \int_0^\pi d\theta\, \sin\theta\, \phi(a,\theta)\, P_\ell(\cos\theta)\,. \tag{4.65}$$

Interior Problem:

A related case arises for the *interior* problem, where the potential is specified on a closed spherical surface $r = a$ and one wishes to solve for the potential everywhere inside this surface. Assuming there are no charges inside the sphere (i.e. it is just a vacuum inside), then the coefficients B_ℓ in the general expansion (4.62) must now be zero, since we require that the potential be regular at $r = 0$. The A_ℓ coefficients will now be given by

$$A_\ell = \tfrac{1}{2}(2\ell+1)\, a^{-\ell} \int_0^\pi d\theta\, \sin\theta\, \phi(a,\theta)\, P_\ell(\cos\theta)\,. \tag{4.66}$$

Region Between Two Spherical Surfaces:

A third related example arises if one wishes to solve for ϕ in the free-space region between two concentric spherical surfaces with $r = a$ and $r = b$. Let us assume that $b > a$. Now, both the A_ℓ and B_ℓ coefficients will in general be non-vanishing. The boundary values for ϕ must be specified on both the surfaces, and so one again has the correct total number of equations to solve for all of the coefficients. Thus one will have

$$\begin{aligned} A_\ell a^\ell + B_\ell a^{-\ell-1} &= \tfrac{1}{2}(2\ell+1) \int_0^\pi d\theta\, \sin\theta\, \phi(a,\theta)\, P_\ell(\cos\theta)\,,\\ A_\ell b^\ell + B_\ell b^{-\ell-1} &= \tfrac{1}{2}(2\ell+1) \int_0^\pi d\theta\, \sin\theta\, \phi(b,\theta)\, P_\ell(\cos\theta)\,, \end{aligned} \tag{4.67}$$

which can be solved for all A_ℓ and B_ℓ.

Further examples:

Of course one can easily consider other classes of problem too. For example, one may consider Neumann cases where it is the normal derivative of ϕ, rather than ϕ itself, that is specified on the spherical boundary or boundaries. Consider, for example, the exterior problem in this case. From (4.62), after setting $A_\ell = 0$ for regularity at infinity, one has

$$\frac{\partial\phi(r,\theta)}{\partial r}\Big|_{r=a} = -\sum_{\ell\ge 0}(\ell+1)B_\ell a^{-\ell-2}\, P_\ell(\cos\theta)\,. \tag{4.68}$$

Multiplying by $P_n(\cos\theta)$ and integrating, we therefore find

$$B_\ell = -\frac{(2\ell+1)}{2(\ell+1)}\, a^{\ell+2} \int_0^\pi d\theta\, \sin\theta\, \Big(\frac{\partial\phi(r,\theta)}{\partial r}\Big|_{r=a}\Big)\, P_\ell(\cos\theta)\,. \tag{4.69}$$

Analogous results can be obtained for the interior problem and for the case of the region between two concentric spheres.

4.6 Some useful techniques for azimuthally-symmetric problems

The procedure for solving Laplace's equation described in the previous subsection is straighforward, but evaluating the integrals in order to read off the expansion coefficients in (4.46) can sometimes be a little involved. There is a very useful "trick" which can often be employed to obtain the answer in a more elegant way, with considerably less effort. We describe this, and a couple of further techniques, below.

Solution by off-axis extrapolation:

In an electrostatics problem with azimuthal symmetry, it is often very easy to obtain the expression for the potential along the axis of symmetry by elementary means. Thus, if we make the natural choice and align the z axis along the axis of azimuthal symmetry, this means that $\phi(z)$ is easily calculated. In terms of spherical polar coordinates, points along the z axis correspond to $\theta = 0$ (for the positive z axis), or $\theta = \pi$ (for the negative z axis).

Suppose that we are considering an "exterior" azimuthally-symmetric electrostatics problem, for which we wish to find $\phi(r,\theta)$ in an exterior region, say $r \ge a$. From (4.46), the general such solution to Laplace's equation will take the form

$$\phi(r,\theta) = \sum_{\ell\ge 0} \frac{B_\ell}{r^{\ell+1}}\, P_\ell(\cos\theta)\,. \tag{4.70}$$

On the positive z axis, where $\theta = 0$, we shall therefore have

$$\phi(z) = \phi(z,0) = \sum_{\ell\ge 0} \frac{B_\ell}{z^{\ell+1}}\, P_\ell(1) = \sum_{\ell\ge 0} \frac{B_\ell}{z^{\ell+1}}\,, \tag{4.71}$$

where, in getting to the final expression, we have used the normalisation property $P_\ell(1) = 1$ of the Legendre polynomials.

It is evident, therefore, that if we already know the on-axis expression for $\phi(z)$ from some elementary calculation, then by expanding it in inverse powers of z and comparing terms with (4.71), the coefficients B_ℓ can all be read off. Having found them from the on-axis calculation, we just plug them into (4.70) and thereby we obtain the complete solution for all r and θ.

As an example, consider the problem that we discussed in section (2.4), when illustrating the use of the Dirichlet Green function for the sphere. The potential on the upper hemishpere of a sphere of radius a was taken to be the constant $+V$, and on the lower hemisphere the potential was $-V$. The exact expression for the potential in the region $r > a$ was obtained in the form of the integral (2.32), but this cannot be evaluated explicitly in any useful form. We observed at the time that the expression for ϕ on the (positive) z axis took the form (2.33), i.e.

$$\phi(z) = V\,\left(1 - \frac{z^2 - a^2}{z\sqrt{z^2+a^2}}\right)\,. \tag{4.72}$$

Writing this as

$$\phi(z) = V\,\left[1 - \left(1 - \frac{a^2}{z^2}\right)\left(1 + \frac{a^2}{z^2}\right)^{-1/2}\right]\,, \tag{4.73}$$

it is a simple matter to expand the inverse square root using the binomial theorem, and therby obtain the expansion for $\phi(z)$ as an infinite series in powers of $1/z$:

$$\begin{aligned}
\phi(z) &= V\,\left[1 - \left(1 - \frac{a^2}{z^2}\right)\sum_{n\ge 0}\frac{(-1)^n\,(2n)!}{2^{2n}\,(n!)^2}\,\left(\frac{a}{z}\right)^{2n}\right] \\
&= V\,\sum_{n\ge 1}\frac{(-1)^{n+1}\,(4n-1)(2n-2)!}{2^{2n-1}\,n!\,(n-1)!}\,\left(\frac{a}{z}\right)^{2n} \\
&= \frac{3V}{2}\,\frac{a^2}{z^2} - \frac{7V}{8}\,\frac{a^4}{z^4} + \frac{11V}{16}\,\frac{a^6}{z^6} - \frac{75V}{128}\,\frac{a^8}{z^8} + \cdots\,.
\end{aligned} \tag{4.74}$$

Reading off the coefficients in the expansion (4.74), we can therefore immediately conclude that the general off-axis series for the potential is

$$\begin{aligned}
\phi(r,\theta) &= V\,\sum_{n\ge 1}\frac{(-1)^{n+1}\,(4n-1)(2n-2)!}{2^{2n-1}\,n!\,(n-1)!}\,\left(\frac{a}{r}\right)^{2n}\,P_{2n-1}(\cos\theta) \qquad (4.75)\\
&= \frac{3V}{2}\,\frac{a^2}{r^2}\,P_1(\cos\theta) - \frac{7V}{8}\,\frac{a^4}{r^4}\,P_3(\cos\theta) + \frac{11V}{16}\,\frac{a^6}{r^6}\,P_5(\cos\theta) - \frac{75V}{128}\,\frac{a^8}{r^8}\,P_7(\cos\theta) + \cdots\,.
\end{aligned}$$

The technique of off-axis extrapolation is clearly very powerful. Indeed, it can really be viewed as the default approach that one should try first, when solving electrostatics problems where there is azimuthal symmetry.

Solution by inversion:

It is sometimes the case that one already has the result for the series expansion in, say, the exterior region $r > a$, and one now wishes to find the series expansion in the interior region $r < a$. This can, of course, be done by going back and re-solving the problem from scratch, in the region $r < a$, where now it will be the A_ℓ coefficients in (4.46), rather than the B_ℓ coefficients, that are non-zero. Sometimes, an easier method is to use the following procedure, known as *Inversion*.

Suppose we have already found that the solution in the exterior region is given by

$$\phi(r,\theta) = \sum_{\ell\ge 0} \frac{B_\ell}{r^{\ell+1}}\, P_\ell(\cos\theta)\,, \tag{4.76}$$

where the coefficients B_ℓ are known. On general grounds, we know that the solution in the interior region will be of the general form

$$\phi(r,\theta) = \sum_{\ell\ge 0} A_\ell\, r^\ell\, P_\ell(\cos\theta)\,. \tag{4.77}$$

(We are assuming there are no free charges in the interior region.)

Under the assumption that there are no charges that could give rise to singularities at the $r = a$ interface, it follows that we can use either of the expressions (4.76) or (4.77) at $r = a$ itself, and so equating the two, we find

$$\sum_{\ell\ge 0} \frac{B_\ell}{a^{\ell+1}}\, P_\ell(\cos\theta) = \sum_{\ell\ge 0} A_\ell\, a^\ell\, P_\ell(\cos\theta)\,. \tag{4.78}$$

Since this equation must hold for all θ, and since the Legendre polynomials are linearly independent, it follows that the coefficients of each P_ℓ separately must be equal on the two sides of the equation. Therefore we can deduce that

$$A_\ell = \frac{B_\ell}{a^{2\ell+1}}\,, \tag{4.79}$$

and hence the series expansion (4.77) in the interior region is given by

$$\phi(r,\theta) = \sum_{\ell\ge 0} \frac{B_\ell\, r^\ell}{a^{2\ell+1}}\, P_\ell(\cos\theta)\,, \tag{4.80}$$

in terms of the already-determined coefficients B_ℓ of the exterior region.

On could, of course, equally well apply the inversion procedure the other way around, and instead find the expansion in the exterior region if the expansion in the interior region is already known.

The example we discussed previously, of the potential outside the two hemispheres held at potentials V and $-V$, provides a nice illustration of the method of inversion. The general solution for the potential outside the two hemispheres was found to be given by (4.75). Using the inversion relation (4.79), it therefore follows that all we need to do to obtain the solution inside the two hemispheres is to apply the replacement rule

$$\left(\frac{a}{r}\right)^{\ell+1} \longrightarrow \left(\frac{r}{a}\right)^{\ell} \tag{4.81}$$

to the term in $P_\ell(\cos\theta)$ in (4.75), for each value of ℓ. This therefore gives the interior solution

$$\begin{aligned}\phi(r,\theta) &= V\sum_{n\ge 1}\frac{(-1)^{n+1}\,(4n-1)(2n-2)!}{2^{2n-1}\,n!\,(n-1)!}\left(\frac{r}{a}\right)^{2n-1} P_{2n-1}(\cos\theta) \qquad (4.82)\\ &= \frac{3V}{2}\,\frac{r}{a}\,P_1(\cos\theta) - \frac{7V}{8}\,\frac{r^3}{a^3}\,P_3(\cos\theta) + \frac{11V}{16}\,\frac{r^5}{a^5}\,P_5(\cos\theta) - \frac{75V}{128}\,\frac{r^7}{a^7}\,P_7(\cos\theta) + \cdots\,.\end{aligned}$$

The method of inversion is clearly a very convenient way of solving the interior problem, once the exterior problem has been solved, or *vice versa*. One must, however, be careful about the circumstances under which it can be applied. It is *essential* that the series expansion that is valid in the region $r > a$ should also be valid actually at $r = a$, and likewise that the solution valid for $r < a$ should be valid at $r = a$. This is an issue of *convergence* of the series, and in turn this is related to the question of the *analyticity* of the solution.

In general, the exterior and interior series will be convergent on the surface $r = a$ itself *as long as there are no singularities at* $r = a$. This is true in the example we considered, where the potential is perfectly finite at all points on the $r = a$ surface (it is either $+V$ or $-V$). However, an example where it may *not* be true is if we considered a situation where there was an infinite charge density somewhere on the $r = a$ surface, such as would result from point charges or other kinds of singular charge densities. In fact the singularities need not be so extreme as in this example, and still the method of inversion may fail. For example, a function such as $f(x) = (x - x_0)^{5/2}$ is singular at $x = x_0$, in the sense that it is not analytic there. (The third derivative, and above, do not exist at $x = x_0$, and so $f(x)$ does not admit a Taylor expansion around $x = x_0$.)

In summary, while the method of inversion can sometimes be useful, it must be applied with great care. It is often in practice safer to recalculate from scratch in the inner region (for example, using the off-axis exptrapolation method discussed previously), rather than applying inversion to the solution in the outer region.

4.7 The spherical harmonics

So far in our discussion of solutions of Laplace's equation in spherical polar coordinates, we have focused on situations with azimuthal symmetry, for which the associated Legendre equation (4.10) reduced to the Legendre equation (4.12). Now, we have to restore the integer m, and consider the associated Legendre equation itself.

For convenience, we present again the Associated Legendre Equation:

$$\frac{d}{dx}\Big((1-x^2)\,\frac{dy}{dx}\Big) + \Big(\lambda - \frac{m^2}{1-x^2}\Big)\,y = 0\,. \tag{4.83}$$

Luckily, it turns out that we can construct the relevant solutions of this equation rather simply, in terms of the Legendre polynomials that we have already studied.

To begin, we write $y = (1-x^2)^{m/2}\,w$, and substitute this into (4.83). After simple algebra we find, after extracting an overall factor of $(1-x^2)^{m/2}$, that w must satisfy

$$(1-x^2)\,w'' - 2(m+1)\,x\,w' + [\lambda - m\,(m+1)]\,w = 0\,. \tag{4.84}$$

(We are using a prime to denote differentiation d/dx here.) Now suppose that we have a solution u of the ordinary Legendre equation:

$$(1-x)^2\,u'' - 2x\,u' + \lambda\,u = 0\,. \tag{4.85}$$

Next, we differentiate this m times. Let us use the notation ∂^m as a shorthand for d^m/dx^m. It is useful to note that we have the following lemma, which is just a consequence of Leibnitz' rule for the differentiation of a product, iterated m times:

$$\begin{aligned}\partial^m(f\,g) \quad &= \quad f\,(\partial^m g) + m\,(\partial f)\,(\partial^{m-1} g) + \frac{m(m-1)}{2!}\,(\partial^2 f)\,(\partial^{m-2} g)\\ &\quad + \frac{m(m-1)(m-2)}{3!}\,(\partial^3 f)\,(\partial^{m-3} g) + \cdots\,.\end{aligned} \tag{4.86}$$

We only need the first two or three terms in this expression if we apply it to the products in (4.85), and so we easily find that

$$(1-x^2)\,\partial^{m+2}u - 2(m+1)\,x\,\partial^{m+1}u + [\lambda - m(m+1]\,\partial^m\,u = 0\,. \tag{4.87}$$

Thus we see that setting $w = \partial^m u$, we have constructed a solution of (4.84) in terms of a solution u of the Legendre equation (4.85). The upshot, therefore, is that if we define

$$P_\ell^m(x) \equiv (-1)^m\,(1-x^2)^{m/2}\,\frac{d^m}{dx^m}\,P_\ell(x)\,, \tag{4.88}$$

where $P_\ell(x)$ is a Legendre polynomial, then $P_\ell^m(x)$ will be a solution of the Associated Legendre Equation with $\lambda = \ell\,(\ell+1)$:

$$\frac{d}{dx}\Big((1-x^2)\,\frac{dP_\ell^m}{dx}\Big) + \Big(\ell\,(\ell+1) - \frac{m^2}{1-x^2}\Big)\,P_\ell^m = 0\,. \tag{4.89}$$

Since $P_\ell(x)$ is regular everywhere including $x = \pm 1$, it is clear that $P_\ell^m(x)$ will be too. (Recall that $x = \cos\theta$, so $(1-x^2)^{m/2} = (\sin\theta)^m$.) It is understood here that we are taking the integer m to be non-negative. It is clear that we must have $m \le \ell$ too, since if m exceeds ℓ then the m-fold derivative of the ℓ'th Legendre polynomial (which itself is of degree ℓ) will give zero.

Recall next that we have Rodrigues' formula (4.36), which gives us an expression for $P_\ell(x)$. Substituting this into (4.88), we get the *Generalised Rodrigues Formula*

$$\boxed{P_\ell^m(x) = \frac{(-1)^m}{2^\ell\,\ell!}\,(1-x^2)^{m/2}\,\frac{d^{\ell+m}}{dx^{\ell+m}}\,(x^2-1)^\ell\,.} \tag{4.90}$$

A nice little miracle now occurs: this formula makes sense for negative values of m too, provided that $m \ge -\ell$. Thus we have a construction of Associated Legendre Functions for all integers m in the interval $-\ell \le m \le \ell$.

Looking at the Associated Legendre Equation (4.89), we note that the equation itself is invariant under sending

$$m \longrightarrow -m\,, \tag{4.91}$$

since m appears only as m^2 in the equation. This means that if we take a solution with a given m, then sending m to $-m$ gives us another solution. What is more, only one solution at fixed ℓ and m^2 can be regular at $x = \pm 1$, since the second solution will have logarithmic singularities there (just like we saw for the Legendre functions). Since $P_\ell^m(x)$ and $P_\ell^{-m}(x)$ given by 4.90 are both regular at $x = \pm 1$ (and therefore neither of them can involve the second solution with logarithmic singularities at $x = \pm 1$), it follows that they must be linearly dependent; i.e. $P_\ell^{-m}(x)$ must be some constant multiple of $P_\ell^m(x)$:

$$P_\ell^{-m}(x) = k\,P_\ell^m(x)\,. \tag{4.92}$$

It is easy to determine what the constant k is, by using (4.90). From (4.92) we get

$$\partial^{\ell-m}(x^2-1)^\ell = k\,(1-x^2)^m\,\partial^{\ell+m}(x^2-1)^\ell\,. \tag{4.93}$$

It is good enough just to look at the highest power of x, since we have already argued that

(4.92) must hold, and so all we need to do is to calculate what k is.[12] Thus we get

$$\frac{(2\ell)!}{(\ell+m)!}\, x^{\ell+m} = k\,(-1)^m\, x^{2m}\, \frac{(2\ell)!}{(\ell-m)!}\, x^{\ell-m} \tag{4.94}$$

at the leading order in x, which fixes k and hence establishes that

$$P_\ell^{-m}(x) = (-1)^m \frac{(\ell-m)!}{(\ell+m)!}\, P_\ell^m(x)\,. \tag{4.95}$$

Using this result we can now very easily work out the normalisation integral for the associated Legendre functions $P_\ell^m(x)$. The relevant integral we shall need to evaluate is

$$\int_{-1}^{1} dx\, P_\ell^m(x)\, P_n^m(x)\,. \tag{4.96}$$

(It will become clear later why we have set the upper indices m equal here.) Using the same method as we used for the Legendre polynomials, it is easy to show that (4.96) vanishes unless $\ell = n$. For $\ell = n$, we can make use of (4.95) to write the required integral as

$$C_{\ell m} \equiv \int_{-1}^{1} dx\, [P_\ell^m(x)]^2 = (-1)^m \frac{(\ell+m)!}{(\ell-m)!} \int_{-1}^{1} dx\, P_\ell^m(x)\, P_\ell^{-m}(x)\,. \tag{4.97}$$

Our task is to calculate the constants $C_{\ell m}$.

Using (4.88), and knowing from (4.90) that (4.88) actually makes perfectly good sense for the negative values of m as well as the positive ones,[13] we shall have

$$\begin{aligned}\int_{-1}^{1} dx\, P_\ell^m(x)\, P_\ell^{-m}(x) &= \int_{-1}^{1} dx\, (-1)^m (1-x^2)^{m/2} \partial^m P_\ell(x)\, (-1)^m (1-x^2)^{-m/2} \partial^{-m} P_\ell(x) \\ &= \int_{-1}^{1} dx\, \partial^m P_\ell(x) \partial^{-m} P_\ell(x)\,. \end{aligned} \tag{4.98}$$

We now integrate by parts m times, to push the ∂^m derivatives onto $\partial^{-m} P_\ell(x)$, noting that the boundary terms will cancel for each integration by parts. (Bear in mind the precise definition of $\partial^{-m} P_\ell(x)$, as explained in footnote 13.) Thus we have

$$\int_{-1}^{1} dx\, P_\ell^m(x)\, P_\ell^{-m}(x) = (-1)^m \int_{-1}^{1} dx (P_\ell(x))^2 = \frac{2(-1)^m}{2\ell+1}\,, \tag{4.99}$$

where we have used the previous result (4.55) in getting to the final result. Looking back to (4.97), we have therefore established that

$$\int_{-1}^{1} dx\, P_\ell^m(x)\, P_{\ell'}^m(x) = \frac{2}{(2\ell+1)}\, \frac{(\ell+m)!}{(\ell-m)!}\, \delta_{\ell\ell'}\,. \tag{4.100}$$

[12] One could, more adventurously, give another proof that $P_\ell^{-m}(x)$ and $P_\ell^m(x)$ are linearly dependent by checking *all* powers of x. We leave this as an exercise for the reader.

[13] $\partial^{-m} P_\ell(x)$ means the m'th indefinite integral of $P_\ell(x)$. Each integration will bring in an associated integration constant, and so in general $\partial^{-m} P_\ell(x)$ would have m arbitrary constants of integration, in the form of coefficients in a polynomial in x of degree $(m-1)$. When we write $\partial^{-m} P_\ell(x)$, it is to be understood that we mean *precisely* $(2^{-\ell}/\ell!)\, \partial^{\ell-m}(x^2-1)^\ell$; in other words, this *defines* the choices we are making for the m integration constants.

Recalling that in the separation of variables discussed in section 4 we had considered factorised solutions of the form $\phi(r,\theta,\varphi) = r^{-1}\,R(r)\,Y(\theta,\varphi)$, and that $Y(\theta,\varphi)$ was itself factorised in the form $\Theta(\theta)\Phi(\varphi)$ with $\Phi(\varphi) \sim e^{\mathrm{i}m\varphi}$ and $\Theta(\theta)$ satisfying the associated Legendre equation, we see that $Y(\theta,\varphi)$ will be of the general form $P_\ell^m(\cos\theta)e^{\mathrm{i}m\varphi}$. To be precise, we shall define

$$Y_{\ell m}(\theta,\varphi) \equiv \sqrt{\frac{(2\ell+1)}{4\pi}}\,\sqrt{\frac{(\ell-m)!}{(\ell+m)!}}\,P_\ell^m(\cos\theta)\,e^{\mathrm{i}\,m\,\varphi}\,,\qquad \ell\ge 0\,,\quad -\ell\le m\le\ell\,. \tag{4.101}$$

The *Spherical Harmonics* $Y_{\ell m}(\theta,\varphi)$ satisfy

$$-\nabla^2_{(\theta,\varphi)}\,Y_{\ell m}(\theta,\varphi) = \ell\,(\ell+1)\,Y_{\ell m}(\theta,\varphi)\,. \tag{4.102}$$

These spherical harmonics form the complete set of regular solutions of $\nabla^2_{(\theta,\varphi)}\,Y = -\lambda\,Y$ on the unit sphere. Note from (4.95) that we have

$$Y_{\ell,-m}(\theta,\varphi) = (-1)^m\,\bar Y_{\ell m}(\theta,\varphi)\,, \tag{4.103}$$

where the bar denotes complex conjugation.

We can easily see that the spherical harmonics satisfy the orthogonality properties

$$\int d\Omega\,\bar Y_{\ell' m'}(\theta\,\varphi)\,Y_{\ell m}(\theta,\varphi) = \delta_{\ell\ell'}\,\delta_{mm'}\,, \tag{4.104}$$

where

$$d\Omega \equiv \sin\theta\,d\theta\,d\varphi \tag{4.105}$$

is the area element on the unit sphere, and $\int d\Omega\,X$ means

$$\int_0^{2\pi} d\varphi\int_0^\pi \sin\theta\,d\theta\,X\,. \tag{4.106}$$

Thus (4.104) just says that the integral on the left-hand side is zero unless $\ell'=\ell$ and $m'=m$. Note that it is the integration over ϕ that is responsible for producing the Kronecker delta $\delta_{mm'}$, since the ϕ dependent factors in (4.104) are

$$\int_0^{2\pi} d\varphi\,e^{\mathrm{i}\,(m-m')\,\varphi}\,. \tag{4.107}$$

This integrates to zero if the integers m and m' are unequal, whilst giving 2π if $m=m'$. The remaining integration over θ in (4.104) then reduces, with m and m' equal, to the integral in (4.100), which then gives rise to the Kronecker delta function $\delta_{\ell\ell'}$ in (4.104).

It is instructive to look at the first few spherical harmonics explicitly. From (4.101), and using (4.90) to give the expressions for the P_ℓ^m, we find

$$
\begin{aligned}
Y_{0,0}(\theta,\varphi) &= \frac{1}{\sqrt{4\pi}}\,,\\[2ex]
Y_{1,1}(\theta,\varphi) &= -\sqrt{\frac{3}{8\pi}}\,\sin\theta\, e^{\mathrm{i}\,\varphi}\,,\\
Y_{1,0}(\theta,\varphi) &= \sqrt{\frac{3}{4\pi}}\,\cos\theta\,,\\
Y_{1,-1}(\theta,\varphi) &= \sqrt{\frac{3}{8\pi}}\,\sin\theta\, e^{-\mathrm{i}\,\varphi}\,,\\[2ex]
Y_{2,2}(\theta,\varphi) &= \sqrt{\frac{15}{32\pi}}\,\sin^2\theta\, e^{2\mathrm{i}\,\varphi}\,,\\
Y_{2,1}(\theta,\varphi) &= -\sqrt{\frac{15}{8\pi}}\,\sin\theta\,\cos\theta\, e^{\mathrm{i}\,\varphi}\,,\\
Y_{2,0}(\theta,\varphi) &= \sqrt{\frac{5}{16\pi}}\,(3\cos^2\theta-1)\,,\\
Y_{2,-1}(\theta,\varphi) &= \sqrt{\frac{15}{8\pi}}\,\sin\theta\,\cos\theta\, e^{-\mathrm{i}\,\varphi}\,,\\
Y_{2,-2}(\theta,\varphi) &= \sqrt{\frac{15}{32\pi}}\,\sin^2\theta\, e^{-2\mathrm{i}\,\varphi}\,.
\end{aligned}
\tag{4.108}
$$

It is also instructive to rewrite the spherical harmonics in terms of Cartesian, rather than spherical polar, coordinates. Recall that the two coordinate systems are related by

$$
x = r\,\sin\theta\,\cos\varphi\,,\qquad y = r\,\sin\theta\,\sin\varphi\,,\qquad z = r\,\cos\theta\,. \tag{4.109}
$$

We can write the expressions for x and y more succinctly in a single complex equation,

$$
x + \mathrm{i}\,y = r\,\sin\theta\, e^{\mathrm{i}\,\varphi}\,, \tag{4.110}
$$

since we have the well-known result that $e^{\mathrm{i}\,\varphi} = \cos\varphi + \mathrm{i}\,\sin\varphi$. Thus for the spherical harmonics listed in (4.108) we have

$$
\begin{aligned}
Y_{0,0} &= \frac{1}{\sqrt{4\pi}}\,,\\
Y_{1,1} &= -\sqrt{\frac{3}{8\pi}}\,\frac{x+\mathrm{i}\,y}{r}\,,\\
Y_{1,0} &= \sqrt{\frac{3}{4\pi}}\,\frac{z}{r}\,,\\
Y_{1,-1} &= \sqrt{\frac{3}{8\pi}}\,\frac{x-\mathrm{i}\,y}{r}\,,\\
Y_{2,2} &= \sqrt{\frac{15}{32\pi}}\,\frac{(x+\mathrm{i}\,y)^2}{r^2}\,,
\end{aligned}
$$

$$
\begin{aligned}
Y_{2,1} &= -\sqrt{\frac{15}{8\pi}}\,\frac{z\,(x+\mathrm{i}\,y)}{r^2}\,,\\
Y_{2,0} &= \sqrt{\frac{5}{16\pi}}\,\Big(\frac{3z^2}{r^2}-1\Big) = \sqrt{\frac{5}{16\pi}}\,\frac{2z^2-x^2-y^2}{r^2}\,,\\
Y_{2,-1} &= \sqrt{\frac{15}{8\pi}}\,\frac{z\,(x-\mathrm{i}\,y)}{r^2}\,,\\
Y_{2,-2} &= \sqrt{\frac{15}{32\pi}}\,\frac{(x-\mathrm{i}\,y)^2}{r^2}\,.
\end{aligned}
\tag{4.111}
$$

What we are seeing here is that for each value of ℓ, we are getting a set of functions, labelled by m with $-\ell \le m \le \ell$, that are all of the form of polynomials of degree ℓ in (x, y, z), divided by r^ℓ:

$$
Y_{\ell m} \sim \frac{x_{i_1}\, x_{i_2} \cdots x_{i_\ell}}{r^\ell}\,. \tag{4.112}
$$

(Here x_i, with $1 \le i \le 3$, denotes the three Cartesian coordinates; $x_1 = x$, $x_2 = y$, $x_3 = z$.) The larger ℓ is, the larger the number of possible such polynomials. Looking at $\ell = 1$, we have in total three $Y_{1,m}$ functions, which could be reorganised, by taking appropriate linear combinations, as

$$
\frac{x}{r}\,,\qquad \frac{y}{r}\,,\qquad \frac{z}{r}\,. \tag{4.113}
$$

Thus once we know one of them, the other two just correspond to rotating the coordinate system through 90 degrees about one or another axis. The same is true of all the higher harmonics too. The spherical harmonics thus have built into them the "knowledge" of the rotational symmetries of the sphere. Our procedure for deriving the spherical harmonics was completely "non-transparent," in the sense that no explicit use of the rotational symmetry of the sphere was made in the derivation. But at the end of the day, we see that the harmonics we have obtained do indeed reflect the symmetries. In the language of group theory, one says that the spherical harmonics $Y_{\ell m}$ fall into representations of the rotation group. One of the rather remarkable "miracles" that we encountered during our derivation, namely that the solutions to the associated Legendre equation could be constructed from solutions of the ordinary Legendre equation, ultimately has its explanation in the fact that the harmonics $Y_{\ell m}$ with $m \ne 0$ are simply related to the $m = 0$ harmonic $Y_{\ell 0}$ by symmetry rotations of the sphere.

4.8 General solution of Laplace's equation without azimuthal symmetry

We have now established that the most general solution of Laplace's equation in spherical polar coordinates[14] can be written as

$$\boxed{\phi(r,\theta,\varphi) = \sum_{\ell\ge 0}\sum_{m=-\ell}^{\ell} (A_{\ell m}\, r^\ell + B_{\ell m}\, r^{-\ell-1})\, Y_{\ell m}(\theta,\varphi)\,. \qquad (4.114)}$$

The constants $A_{\ell m}$ and $B_{\ell m}$, which depend on both ℓ and m, are as yet arbitrary. Their values are determined by boundary conditions, as in the previous potential-theory examples that we have looked at. Because we are now allowing the azimuthal separation constant m to be non-zero, the class of solutions described by (4.114) includes those that are dependent on the azimuthal angle φ.

In a boundary-value problem where the potential $\phi(r,\theta,\varphi)$ is specified on a spherical boundary surface at $r = a$, one simply uses the orthogonality conditions (4.104) in order to determine the coefficients $A_{\ell m}$ and $B_\ell m$ in the general solution (4.114). For example, suppose we are solving for ϕ in the exterior region $r > a$. This means that $A_{\ell m} = 0$, and so, setting $r = a$, multiplying by $\bar Y_{\ell',m'}(\theta,\varphi)$ and integrating, we shall have

$$\begin{aligned} \int \phi(a,\theta,\varphi)\bar Y_{\ell',m'}(\theta,\varphi) d\Omega &= \sum_{\ell\ge 0}\sum_{m=-\ell}^{\ell} B_{\ell m}\, a^{-\ell-1} \int Y_{\ell m}(\theta,\varphi)\bar Y_{\ell',m'}(\theta,\varphi) d\Omega \\ &= \sum_{\ell\ge 0}\sum_{m=-\ell}^{\ell} B_{\ell m}\, a^{-\ell-1}\, \delta_{\ell,\ell'}\, \delta_{m,m'} \\ &= B_{\ell',m'}\, a^{-\ell'-1}\,, \qquad (4.115) \end{aligned}$$

and so we have

$$\boxed{B_{\ell m} = a^{\ell+1} \int \phi(a,\theta,\varphi)\bar Y_{\ell m}(\theta,\varphi) d\Omega\,. \qquad (4.116)}$$

The specification of an electrostatics problem is not always given in the convenient form of boundary values on a spherical surface. Consider, for example, our discussion of the Green function for an infinite planar surface, in section 2.2. Equation (2.6) gives the expression for $\phi(x,y,z)$ everywhere in the half-space $z > 0$ in terms of its boundary value on the plane $z = 0$. Suppose we take the boundary potential to be equal to 1 within the square of side 2 whose vertices are located at $x = \pm 1$, $y = \pm 1$, and take the potential to be zero everywhere outside this square in the $z = 0$ plane. Thus, the potential for $z > 0$ is

[14] That is, the most general solution that is regular on the spherical surfaces at constant r.

given by

$$\phi(x,y,z) = \frac{z}{2\pi} \int_{-1}^{1} dx' \int_{-1}^{1} dy' \frac{\phi(x',y',0)}{[(x-x')^2+(y-y')^2+z^2]^{3/2}} \,. \tag{4.117}$$

Although this can be integrated explicitly, the result is rather complicated. Instead, let us write x, y and z in terms of spherical polar coordinates, expand the integrand in inverse powers of r, and integrate term by term. After some algebra, which is easily done using *Mathematica*, we find

$$\begin{aligned} \phi(r,\theta,\varphi) &= \frac{2c}{\pi r^2} + \frac{3c-5c^3}{\pi r^4} + \frac{7c(15-70c^2+63c^4-9s^4\cos 4\varphi)}{24\pi r^6} \\ &\quad + \frac{3c(35-315c^2+693c^4-429c^6+11s^4(13c^2-3)\cos 4\varphi)}{16\pi r^8} + \cdots \,, \end{aligned} \tag{4.118}$$

where we have defined $c = \cos\theta$ and $s = \sin\theta$. From the definition of $Y_{\ell m}(\theta,\varphi)$, it is not hard to express (4.118) as a series in the spherical harmonics, giving

$$\begin{aligned} \phi(r,\theta,\varphi) &= \frac{4}{\sqrt{3\pi}}\frac{1}{r^2}\, Y_{1,0}(\theta,\varphi) - \frac{4}{\sqrt{7\pi}}\frac{1}{r^4}\, Y_{3,0}(\theta,\varphi) \\ &\quad + \frac{1}{r^6}\left(\frac{14}{3\sqrt{11\pi}} Y_{5,0}(\theta,\varphi) - \sqrt{\frac{14}{55\pi}}(Y_{5,4}(\theta,\varphi)+Y_{5,-4}(\theta,\varphi))\right) \\ &\quad + \frac{1}{r^8}\left(-2\sqrt{\frac{3}{5\pi}} Y_{7,0}(\theta,\varphi) + \sqrt{\frac{22}{35\pi}}(Y_{7,4}(\theta,\varphi)+Y_{7,-4}(\theta,\varphi))\right) + \cdots \end{aligned} \tag{4.119}$$

Observe that the expression for the potential is azimuthally symmetric at the first couple of orders in inverse powers of $1/r$. The non-azimuthal symmetry, which must of course be present since the original boundary value specification involves a square of non-zero potential at $z=0$, sets in at orders $1/r^6$ and above.

Observe also that the potential exhibits the four-fold symmetry that we should expect, given that the boundary value specification itself has a four-fold rotational symmetry. In other words, the potential must exhibit the property of quarter-rotational symmetry:

$$\phi(r,\theta,\varphi) = \phi(r,\theta,\varphi+\tfrac{1}{2}\pi) \,. \tag{4.120}$$

Note that this means that the answer *must* involve only those spherical harmonics $Y_{\ell m}(\theta,\varphi)$ for which $Y_{\ell m}(\theta,\varphi) = Y_{\ell m}(\theta,\varphi+\frac{1}{2}\pi)$, and hence

$$m = 4n\,, \qquad n = \text{ integer}\,. \tag{4.121}$$

Since we have $|m| \le \ell$, this means that it would be *impossible* for any azimuthal dependence to appear before order $1/r^5$ at the absolute earliest. In fact, because other symmetries of the problem mean that only *even* inverse powers of r occur here, the azimuthal dependence is deferred until order $1/r^6$.

4.9 Another look at the generating function

We now return to the generating function for the Legendre polynomials, defined in (4.39). There is a nice physical interpretation of this construction, which we shall now describe.

Consider the problem of a point charge of unit strength, sitting on the z axis at a point $z = r'$. We know, since it is an axially-symmetric situation, that the potential at an arbitrary point must be expressible in the form (4.62)

$$\phi(r,\theta) = \sum_{\ell\ge 0} (A_\ell\, r^\ell + B_\ell\, r^{-\ell-1})\, P_\ell(\cos\theta)\,. \tag{4.122}$$

To determine the coefficients, we must first make a choice between considering either the region where $r > r'$, or the region where $r < r'$.

For $r > r'$, the solution should be an infinite series in inverse powers of r, so that it dies off at infinity. Thus for $r > r'$ we must have $A_\ell = 0$. On the other hand, for $r < r'$ the solution should be an infinite series in positive powers of r, so that it remains regular at $r = 0$. For $r < r'$, therefore, we must have $B_\ell = 0$.

The non-zero coefficients B_ℓ or A_ℓ in the two regions can be determined by the method we discussed earlier, of first finding the potential when $\vec r$ is on the z axis. This is easy; we shall have

$$\begin{aligned} r > r' : \qquad \phi &= \frac{1}{r - r'} = \frac{1}{r}\Big(1 - \frac{r'}{r}\Big)^{-1} = \sum_{\ell\ge 0} \frac{r'^\ell}{r^{\ell+1}}\,,\\ r < r' : \qquad \phi &= \frac{1}{r' - r} = \frac{1}{r'}\Big(1 - \frac{r}{r'}\Big)^{-1} = \sum_{\ell\ge 0} \frac{r^\ell}{r'^{\ell+1}}\,, \end{aligned} \tag{4.123}$$

The general off-axis solution, where $\vec r$ is arbitrary, is therefore given by

$$\boxed{\begin{aligned} \phi(r,\theta) &= \sum_{\ell\ge 0} \frac{r'^\ell}{r^{\ell+1}}\, P_\ell(\cos\theta)\,, \qquad r > r'\,,\\ \phi(r,\theta) &= \sum_{\ell\ge 0} \frac{r^\ell}{r'^{\ell+1}}\, P_\ell(\cos\theta)\,, \qquad r < r'\,. \end{aligned}} \tag{4.124}$$

These expressions give the potential at a point (r,θ,φ) due to a unit point charge at the point $z = r'$ on the z axis. The answer is, of course, independent of the azimuthal angle φ because the point charge is on the z axis.

Note that it is sometimes convenient to write the two expressions in (4.124) in the form of a single equation

$$\boxed{\phi(r,\theta) = \sum_{\ell\ge 0} \frac{r_<^\ell}{r_>^{\ell+1}}\, P_\ell(\cos\theta)\,,} \tag{4.125}$$

where $r_<$ means whichever of r and r' is the smaller, and $r_>$ means whichever of r and r' is the larger.

We can relate these results to the generating function, by observing that we can in fact write down the solution to this problem in closed form. The potential $\phi(r,\theta)$ will just be the inverse of the distance from the point (r,θ) to the point $z = r'$ on the z axis where the unit charge is located. Using the cosine rule, this distance is $(r^2 - 2r\,r'\,\cos\theta + r'^2)^{1/2}$, and so

$$\phi(r,\theta) = \frac{1}{(r^2 - 2r\,r'\,\cos\theta + r'^2)^{1/2}}\,. \tag{4.126}$$

When $r > r'$, this can be expanded in powers of r'/r. Letting $t = r'/r$, we therefore have

$$\phi(r,\theta) = \frac{1}{r}\,\frac{1}{(1 - 2t\,\cos\theta + t^2)^{1/2}}\,, \tag{4.127}$$

and recalling the generating function formula (4.39), we see that this gives

$$\phi(r,\theta) = \frac{1}{r}\sum_{\ell\ge0} t^\ell P_\ell(\cos\theta) = \sum_{\ell\ge0} \frac{r'^\ell}{r^{\ell+1}}\,P_\ell(\cos\theta)\,. \tag{4.128}$$

This agrees precisely with the $r > r'$ expansion we obtained in (4.124).

It is straightforward to repeat the above exercise for the region where $r < r'$. This time, one must pull out a factor of r' in the denominator of (4.126), and expand it in powers of $t = r/r'$, to give, using (4.39),

$$\phi(r,\theta) = \frac{1}{r'}\sum_{\ell\ge0} t^\ell\,P_\ell(\cos\theta) = \sum_{\ell\ge0} \frac{r^\ell}{r'^{\ell+1}}\,P_\ell(\cos\theta)\,, \tag{4.129}$$

and this agrees precisely with the $r < r'$ series we obtained in (4.124).

The discussions above show how the generating function (4.39) admits a very simple physical interpretation as giving the expansion, in terms of Legendre polynomials, of the potential due to a unit point charge on the z axis.

We may also generalise the discussion, to the case where the unit charge is placed at a general position (r',θ',φ') that need not lie on the z axis. The expression for the potential at (r,θ,φ) is therefore no longer azimuthally symmetric, and so it must be expanded in the general form (4.114), in terms of spherical harmonics. Of course we can expand it simply using Legendre polynomials by introducing the angle γ between $\vec r$ and $\vec r\,'$, so that we may write

$$\phi(r,\theta,\varphi) = (r^2 + r'^2 - 2rr'\,\cos\gamma)^{-1/2}\,. \tag{4.130}$$

Using the generating function formula (4.39) we may therefore write

$$\boxed{\begin{aligned} \frac{1}{|\vec r-\vec r\,'|} &= \sum_{\ell\ge0}\frac{r'^\ell}{r^{\ell+1}}\,P_\ell(\cos\gamma)\,, \qquad r>r'\,,\\ \frac{1}{|\vec r-\vec r\,'|} &= \sum_{\ell\ge0}\frac{r^\ell}{r'^{\ell+1}}\,P_\ell(\cos\gamma)\,, \qquad r<r'\,. \end{aligned}} \tag{4.131}$$

However, as noted earlier the expression for γ in terms of θ, φ, θ' and φ' is actually rather complicated (see (2.29)).

To obtain the expression for $|\vec r-\vec r\,'|^{-1}$ in terms of a series in spherical harmonics, we first note from (4.114) that in the region $r>r'$ we shall have

$$\frac{1}{|\vec r-\vec r\,'|}=\sum_{\ell\ge0}\sum_{m=-\ell}^{\ell}\frac{1}{r^{\ell+1}}\,B_{\ell m}(r',\theta',\varphi')\,Y_{\ell m}(\theta,\varphi)\,, \tag{4.132}$$

where we have indicated explicitly that the "constants" $B_{\ell m}(r',\theta',\varphi')$ will depend upon the chosen location (r',θ',φ') for the unit charge. On the other hand, in the region $r<r'$ we shall have

$$\frac{1}{|\vec r-\vec r\,'|}=\sum_{\ell\ge0}\sum_{m=-\ell}^{\ell}r^\ell\,A_{\ell m}(r',\theta',\varphi')\,Y_{\ell m}(\theta,\varphi)\,. \tag{4.133}$$

How do we determine the expansion coefficients $A_{\ell m}(r',\theta',\varphi')$ and $B_{\ell m}(r',\theta',\varphi')$? The first thing to notice is that the function $|\vec r-\vec r\,'|^{-1}$ itself is symmetrical under the exchange of $\vec r$ and $\vec r\,'$. This means that the way in which r', θ' and φ' enter into the expansions (4.132) and (4.133) should be symmetrically related to the way in which r, θ and φ enter. In saying this, due allowance must be made for the fact that if $r>r'$, then, of course $r'<r$, so the way in which r' appears in (4.132) should be very like the way that r appears in (4.133), and *vice versa*. At this point, we shall write down the answer, and then prove it afterwards. The result is that

$$\boxed{\begin{aligned} \frac{1}{|\vec r-\vec r\,'|} &= \sum_{\ell\ge0}\sum_{m=-\ell}^{\ell}\frac{4\pi}{2\ell+1}\,\frac{r'^\ell}{r^{\ell+1}}\,\bar Y_{\ell m}(\theta',\varphi')\,Y_{\ell m}(\theta,\varphi)\,, \qquad r>r'\,, \qquad (4.134)\\ \frac{1}{|\vec r-\vec r\,'|} &= \sum_{\ell\ge0}\sum_{m=-\ell}^{\ell}\frac{4\pi}{2\ell+1}\,\frac{r^\ell}{r'^{\ell+1}}\,\bar Y_{\ell m}(\theta',\varphi')\,Y_{\ell m}(\theta,\varphi)\,, \qquad r<r'\,, \qquad (4.135) \end{aligned}}$$

This result can also be written as the single equation

$$\boxed{\frac{1}{|\vec r-\vec r\,'|}=\sum_{\ell\ge0}\sum_{m=-\ell}^{\ell}\frac{4\pi}{2\ell+1}\,\frac{r_<^\ell}{r_>^{\ell+1}}\,\bar Y_{\ell m}(\theta',\varphi')\,Y_{\ell m}(\theta,\varphi)\,.} \tag{4.136}$$

To prove this, we shall need to establish a couple of preliminary results. First, we note that since the spherical harmonics form, by construction, a complete set of functions on the unit sphere, we can expand an arbitrary smooth function $f(\theta,\varphi)$ defined on the sphere in the form of a generalised Fourier expansion

$$f(\theta,\varphi)=\sum_{\ell\ge 0}\sum_{m=-\ell}^{\ell} b_{\ell m}\, Y_{\ell m}(\theta,\varphi)\,. \tag{4.137}$$

Using the orthonormality relation (4.104) for the spherical harmonics, we find

$$b_{\ell m}=\int d\Omega \bar Y_{\ell m}(\theta,\varphi)\, f(\theta,\varphi)\,, \tag{4.138}$$

and hence substituting this back into (4.137) we obtain

$$f(\theta,\varphi)=\int d\Omega' f(\theta',\varphi')\,\Big(\sum_{\ell\ge 0}\sum_{m=-\ell}^{\ell} \bar Y_{\ell m}(\theta',\varphi') Y_{\ell m}(\theta,\varphi)\Big)\,. \tag{4.139}$$

This implies the completeness relation

$$\boxed{\sum_{\ell\ge 0}\sum_{m=-\ell}^{\ell} \bar Y_{\ell m}(\theta',\varphi') Y_{\ell m}(\theta,\varphi)=\frac{\delta(\theta-\theta')\delta(\varphi-\varphi')}{\sin\theta'}\,.} \tag{4.140}$$

(The $\sin\theta'$ in the denominator is needed to cancel the $\sin\theta'$ factor in the area element $d\Omega'=\sin\theta'\, d\theta' d\varphi'$.)

Secondly, it is useful to introduce the *Heaviside function* $\vartheta(x)$, whose definition is

$$\begin{aligned} \vartheta(x) &= 0\,, \qquad x<0\,,\\ \vartheta(x) &= 1\,, \qquad x>0\,, \end{aligned} \tag{4.141}$$

The discontinuous jump as x passes through zero implies that there is an infinite spike in the derivative of $\vartheta(x)$ at $x=0$, and in fact

$$\vartheta'(x)=\delta(x)\,. \tag{4.142}$$

This can be verified by integrating $\vartheta'(x)$ over the interval $x_1\le x\le x_2$:

$$\int_{x_1}^{x_2} \vartheta'(x) dx=\Big[\vartheta(x)\Big]_{x_1}^{x_2}\,. \tag{4.143}$$

From the definition (4.141), we see that this is equal to 1 if $x=0$ lies in the integration range, but it is instead 0 if $x=0$ lies outside the integration range. The integral of $\delta(x)$ has exactly the same features.

Using the Heaviside function, the two expressions (4.134) and (4.135) for $|\vec{r}-\vec{r}\,'|^{-1}$ can be combined into the single formula:

$$\boxed{\frac{1}{|\vec{r}-\vec{r}\,'|} = \sum_{\ell\ge 0}\sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1}\,\Big(\vartheta(r'-r)\,\frac{r^\ell}{r'^{\ell+1}} + \vartheta(r-r')\,\frac{r'^\ell}{r^{\ell+1}}\Big)\,\bar Y_{\ell m}(\theta',\varphi')\,Y_{\ell m}(\theta,\varphi)\,. \qquad (4.144)}$$

We are now in a position to verify that (4.144) is indeed correct, by verifying that it does indeed satisfy the proper equation

$$\nabla^2\frac{1}{|\vec{r}-\vec{r}\,'|} = -4\pi\delta^3(\vec{r}-\vec{r}\,')\,. \qquad (4.145)$$

(This, together with the fact that the right-hand side of (4.144) obviously satisfies the proper boundary condition that it goes to zero as r or r' goes to infinity, pins down the Green function $|\vec{r}-\vec{r}\,'|^{-1}$ uniquely.)

Viewed as a function of r, θ and φ, the right-hand side of (4.144) *by construction* is annihilated by the Laplacian in the bulk, i.e. away form the crossover at $r=r'$. This is obvious from the fact that in each region $r>r'$ and $r<r'$, it is a particular instance of an infinite series of the form (4.114), which by construction is a solution of Laplace's equation. Therefore, when we act with ∇^2 on the right-hand side of (4.144), all the terms involving θ and φ derivatives, together with those from the r derivatives that *do not act on the Heaviside functions*, will add to zero. We therefore need only to retain those terms in which the r derivatives act on the Heaviside functions.

To save some writing, let us temporarily define

$$W_{\ell m} \equiv \frac{4\pi}{2\ell+1}\,\bar Y_{\ell m}(\theta',\varphi')\,Y_{\ell m}(\theta,\varphi)\,. \qquad (4.146)$$

Recalling that the radial part of $\nabla^2\phi$ can be written as $r^{-1}\partial^2(r\phi)/\partial r^2$, we shall have

$$\begin{aligned}
&\frac{1}{r}\frac{\partial^2}{\partial r^2}\sum_{\ell\ge 0}\sum_{m=-\ell}^{\ell} W_{\ell m}\,\Big(\vartheta(r'-r)\,\frac{r^{\ell+1}}{r'^{\ell+1}} + \vartheta(r-r')\,\frac{r'^\ell}{r^\ell}\Big)\\
&= \frac{1}{r}\frac{\partial}{\partial r}\sum_{\ell\ge 0}\sum_{m=-\ell}^{\ell} W_{\ell m}\,\Big(\vartheta'(r'-r)\,\frac{r^{\ell+1}}{r'^{\ell+1}} + (\ell+1)\vartheta(r'-r)\frac{r^\ell}{r'^{\ell+1}} + \vartheta'(r-r')\,\frac{r'^\ell}{r^\ell} - \ell\vartheta(r-r')\frac{r'^\ell}{r^{\ell+1}}\Big)\\
&= \frac{1}{r}\frac{\partial}{\partial r}\sum_{\ell\ge 0}\sum_{m=-\ell}^{\ell} W_{\ell m}\,\Big(-\delta(r-r')\,\frac{r^{\ell+1}}{r'^{\ell+1}} + (\ell+1)\vartheta(r'-r)\frac{r^\ell}{r'^{\ell+1}} + \delta(r-r')\,\frac{r'^\ell}{r^\ell} - \ell\vartheta(r-r')\frac{r'^\ell}{r^{\ell+1}}\Big)\\
&= \frac{1}{r}\frac{\partial}{\partial r}\sum_{\ell\ge 0}\sum_{m=-\ell}^{\ell} W_{\ell m}\,\Big((\ell+1)\vartheta(r'-r)\,\frac{r^\ell}{r'^{\ell+1}} - \ell\vartheta(r-r')\,\frac{r'^\ell}{r^{\ell+1}}\Big)
\end{aligned}$$

$$
= \sum_{\ell\ge 0}\sum_{m=-\ell}^{\ell} W_{\ell m}\Big[\ell(\ell+1)\Big(\vartheta(r'-r)\,\frac{r^{\ell-1}}{r'^{\ell+1}} + \vartheta(r-r')\frac{r'^{\ell}}{r^{\ell+2}}\Big)
$$

$$
+(\ell+1)\theta'(r'-r)\frac{r^{\ell}}{r'^{\ell+1}} - \ell\vartheta'(r-r')\,\frac{r'^{\ell}}{r^{\ell+1}}\Big]\,. \tag{4.147}
$$

(Note that $\vartheta'(r'-r) = -\delta(r-r')$. Of course one should be careful to distinguish between the two totally different usages of the "prime" symbol, with a prime on a function such as $\vartheta(r-r')$ denoting the derivative with respect to r, while a prime on the coordinates (r', θ', φ') denotes the location of the charge, as opposed to the located (r,θ,φ) of the observation point!)

The terms on the first line of the last expression here are just the usual "bulk" terms, with no derivatives on the Heaviside functions, which will cancel againts the terms from the angular derivatives in ∇^2, as we mentioned above. It is the second line of the last expression that we are after. This second line then becomes

$$
\begin{aligned}
&= \frac{1}{r}\sum_{\ell\ge 0}\sum_{m=-\ell}^{\ell} W_{\ell m}\left(-(\ell+1)\delta(r-r')\frac{r^{\ell}}{r'^{\ell+1}} - \ell\delta(r-r')\frac{r'^{\ell}}{r^{\ell+1}}\right)\\
&= -\frac{1}{r^2}\sum_{\ell\ge 0}\sum_{m=-\ell}^{\ell} W_{\ell m}(2\ell+1)\delta(r-r')\\
&= -\frac{4\pi}{r^2}\,\delta(r-r')\sum_{\ell\ge 0}\sum_{m=-\ell}^{\ell} \bar Y_{\ell m}(\theta',\varphi')\,Y_{\ell m}(\theta,\varphi)\\
&= -4\pi\,\frac{\delta(r-r')\delta(\theta-\theta')\delta(\varphi-\varphi')}{r^2\sin\theta}\\
&= -4\pi\delta^3(\vec r-\vec r\,')\,.
\end{aligned} \tag{4.148}
$$

Note that in the derivation above a prime on ϑ always means a derivative with respect to r, and we have used the results $\vartheta'(r-r') = \delta(r-r')$ and $\vartheta'(r'-r) = -\delta(r'-r) = -\delta(r-r')$. We have also used the property that since $\delta(r-r')$ is non-zero only when $r=r'$, then in any function of r and r' that is multiplied by $\delta(r-r')$, we can always set $r'=r$. Thus, for example, $\delta(r-r')r'^{\ell}/r^{\ell} = \delta(r-r')$.

To summarise, we have shown that indeed $|\vec r-\vec r\,'|^{-1}$ can be written as (4.134) and (4.135) (or, equivalently, as (4.144)), since we have verified that these infinite sums imply that $|\vec r-\vec r\,'|^{-1}$ does indeed satisfy (4.145), as it should. It is interesting to see how the delta function on the right-hand side of (4.145) arises in this representation of $|\vec r-\vec r\,'|^{-1}$ in (4.144). The factors in the θ and φ directions come from the completeness relation (4.140), but the factor in the radial direction arises quite differently. It comes from the fact that the expressions (4.134) and (4.135) in the $r > r'$ and $r < r'$ regions, which are combined into one formula in (4.144), are themselves *equal* at $r = r'$, but there is a discontinuity in the gradient. This in turn implies that there is a delta function in the second derivative with respect to r, and it is this that produces the radial delta function factor.

Observe also that by comparing the expansions (4.131) with those in (4.134) and (4.135), we can deduce that

$$P_\ell(\cos\gamma) = \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} \bar{Y}_{\ell m}(\theta', \varphi') Y_{\ell m}(\theta, \varphi)\,, \tag{4.149}$$

where, it will be recalled, γ is given by (2.29).

4.10 Dirichlet Green function expansion

The expression (4.144) obtained in the previous section is useful in its own right, in situations where there is a charge distribution $\rho(\vec{r})$ and one wants to calculate the resulting contribution to the potential as given by (1.79) in the case that there are no boundaries.

We can also make use of (4.144) in order to obtain an expansion for the Dirichlet Green function for a spherical boundary. Recall that we used the method of images in order to construct the Dirichlet Green function (2.24) for the exterior boundary value problem in which the potential is specified on the spherical surface $r = a$, and one wants to determine $\phi(r, \theta, \varphi)$ for all $r \ge a$. For convenience, we give the expression (2.24) again here:

$$G_D(\vec{r}, \vec{r}\,') = \frac{1}{|\vec{r} - \vec{r}\,'|} - \frac{a/r'}{|\vec{r} - (a^2/r'^2)\,\vec{r}\,'|}\,. \tag{4.150}$$

It then follows straightforwardly from (4.134) and (4.135) that the expansion for $G_D(\vec{r}, \vec{r}\,')$ in (4.150) will be given by

$$\boxed{\begin{aligned} G_D(\vec{r}, \vec{r}\,') &= \sum_{\ell\ge 0} \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1} \frac{1}{r^{\ell+1}} \left(r'^\ell - \frac{a^{2\ell+1}}{r'^{\ell+1}} \right) \bar{Y}_{\ell m}(\theta', \varphi')\, Y_{\ell m}(\theta, \varphi)\,, \qquad r > r'\,, \\ G_D(\vec{r}, \vec{r}\,') &= \sum_{\ell\ge 0} \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1} \frac{1}{r'^{\ell+1}} \left(r^\ell - \frac{a^{2\ell+1}}{r^{\ell+1}} \right) \bar{Y}_{\ell m}(\theta', \varphi')\, Y_{\ell m}(\theta, \varphi)\,, \qquad r < r'\,, \end{aligned}} \tag{4.151}$$

Recall from our original discussion of the Green function that for the Dirichlet boundary-value problem, the potential inside a volume V bounded by a surface S is given by (1.90)

$$\boxed{\phi(\vec{r}) = \int_V \rho(\vec{r}\,')\, G_D(\vec{r}, \vec{r}\,') d^3\vec{r}\,' - \frac{1}{4\pi} \int_S \phi(\vec{r}\,')\, \vec{\nabla}' G_D(\vec{r}, \vec{r}\,') \cdot d\vec{S}'\,.} \tag{4.152}$$

We therefore need to calculate the normal derivative of the upper expression in (4.151) at $r' = a$ (approached from $r' > a$, since we are solving the exterior Dirichlet problem). The normal derivative should be directed *out* of the volume V, which in our case is all of space in the region $r > a$, and so the normal derivative that we require is $(-\partial G_D(\vec{r}, \vec{r}\,')/\partial r')|_{r'=a}$. From the upper expression in (4.151), this is therefore given by

$$-\frac{\partial G_D(\vec{r}, \vec{r}\,')}{\partial r'}\Big|_{r'=a} = -\frac{4\pi}{a^2} \sum_{\ell \ge 0} \sum_{m=-\ell}^{\ell} \Big(\frac{a}{r}\Big)^{\ell+1} \bar{Y}_{\ell m}(\theta', \varphi') Y_{\ell m}(\theta, \varphi)\,. \tag{4.153}$$

Consider, for simplicity, the case where $\rho = 0$ in the entire region $r > a$. It then follows from (4.152) and (4.153) that $\phi(r, \theta, \varphi)$ outside the surface $r = a$ is given in terms of the boundary values $\phi(a, \theta, \varphi)$ on the sphere by

$$\phi(r, \theta, \varphi) = \sum_{\ell \ge 0} \sum_{m=-\ell}^{\ell} \Big(\int \phi(a, \theta', \varphi') \bar{Y}_{\ell m}(\theta', \varphi') d\Omega'\Big) \Big(\frac{a}{r}\Big)^{\ell+1} Y_{\ell m}(\theta, \varphi)\,. \tag{4.154}$$

This can be seen to be contained, as it must be, within the general class of solutions (4.114). It corresponds to $A_{\ell m} = 0$ and

$$B_{\ell m} = a^{\ell+1} \int \phi(a, \theta', \varphi') \bar{Y}_{\ell m}(\theta', \varphi') d\Omega'\,. \tag{4.155}$$

An analogous discussion can be given for the interior problem, where one solves for $\phi(r, \theta, \varphi)$ for $0 < r < a$ in terms of boundary values $\phi(a, \theta, \varphi)$ on the sphere at $r = a$.

5 Separation of Variables in Cylindrical Polar Coordinates

Another common situation that arises when considering boundary-value problems in electrostatics is when there is a cylindrical symmetry, in which case cylindrical polar coordinates are typically the most convenient choice. We shall take these to be (ρ, φ, z), where

$$x = \rho \cos\varphi\,, \qquad y = \rho \sin\varphi\,, \qquad z = z\,. \tag{5.1}$$

In other words, we still use z as the coordinate along the Cartesian z axis, but in the (x, y) plane we use polar coordinates (ρ, φ). A straighforward calculation shows that Laplace's equation (3.1) becomes

$$\frac{1}{\rho} \frac{\partial}{\partial \rho}\Big(\rho \frac{\partial \phi}{\partial \rho}\Big) + \frac{1}{\rho^2} \frac{\partial^2 \phi}{\partial \varphi^2} + \frac{\partial^2 \phi}{\partial z^2} = 0\,. \tag{5.2}$$

We can separate variables by writing $\phi(\rho, \varphi, z) = R(\rho)\, \Phi(\phi)\, Z(z)$, which leads, after dividing out by ϕ, to

$$\frac{1}{\rho R} \frac{d}{d\rho}\Big(\rho \frac{dR}{d\rho}\Big) + \frac{1}{\rho^2 \Phi} \frac{d^2 \Phi}{d\varphi^2} + \frac{1}{Z} \frac{d^2 Z}{dz^2} = 0\,. \tag{5.3}$$

The first two terms depend on ρ and φ but not z, whilst the last term depends on z but not ρ and φ. Thus the last term must be a constant, which we shall call k^2, and then

$$\frac{1}{\rho\, R}\, \frac{d}{d\rho}\Big(\rho \frac{dR}{d\rho}\Big) + \frac{1}{\rho^2\, \Phi}\, \frac{d^2\Phi}{d\varphi^2} + k^2 = 0\,. \tag{5.4}$$

Multiplying by ρ^2, we obtain

$$\frac{\rho}{R}\, \frac{d}{d\rho}\Big(\rho \frac{dR}{d\rho}\Big) + k^2\, \rho^2 + \frac{1}{\Phi}\, \frac{d^2\Phi}{d\varphi^2} = 0\,. \tag{5.5}$$

The first two terms depend on ρ but not φ, whilst the last term depends on φ but not ρ. We deduce that the last term is a constant, which we shal call $-\nu^2$. The separation process is now complete, and we have

$$\frac{d^2 Z}{dz^2} - k^2\, Z = 0\,, \qquad \frac{d^2\Phi}{d\varphi^2} + \nu^2\, \Phi = 0\,, \tag{5.6}$$

$$\frac{d^2 R}{d\rho^2} + \frac{1}{\rho}\, \frac{dR}{d\rho} + \Big(k^2 - \frac{\nu^2}{\rho^2}\Big)\, R = 0\,, \tag{5.7}$$

where k^2 and ν^2 are separation constants.

The Z equation and Φ equation in (5.6) are easily solved, giving

$$Z(z) \sim e^{\pm k z}\,, \qquad \Phi(\varphi) \sim e^{\pm \mathrm{i} \nu \varphi}\,. \tag{5.8}$$

Usually, we shall be seeking solutions where the potential should be periodic in the polar angle φ in the (x, y) plane, so that $\Phi(\varphi + 2\pi) = \Phi(\varphi)$, and hence

$$\nu = \text{ integer}\,. \tag{5.9}$$

However, it will still be useful to think of ν as being a more general real variable for now. The reason for this will emerge when we investigate the solutions of the radial equation (5.7).

Rescaling the radial coordinate by defining $x = k\,\rho$, and renaming R as y (these functions x and y are not to be confused with the original Cartesian coordinates x and y!), the radial equation (5.7) takes the form

$$x^2\, y''(x) + x\, y'(x) + (x^2 - \nu^2)\, y = 0\,, \tag{5.10}$$

where y' means dy/dx. This is known as *Bessel's Equation*, and we can construct solutions in the form of power-series expansions, by applying Frobenius's method as we did when discussing the Legendre equation.

5.1 Solutions of Bessel's equation

It may be recalled that when seeking power-series solutions of Legendre's equation in the form of expansions around $x = 0$, it was sufficient to consider Taylor expansions in non-negative integer powers of x, since $x = 0$ was an *ordinary point* of the Legendre equation. By contrast, the point $x = 0$ is a *singular point* of the Bessel equation. This can be seen by dividing out (5.10) by x^2 so that the y'' term has unit coefficient, and then noting that the coefficients of y' and y become singular at $x = 0$. Technically, the nature of the behaviour at $x = 0$ implies that it is a *regular singular point*, and the upshot is that we should now seek solutions of the form

$$y(x) = \sum_{n\ge 0} a_n\, x^{n+\sigma}\,, \tag{5.11}$$

where σ is a constant. Substituting into (5.10), we obtain

$$\sum_{n\ge 0} [(n+\sigma)^2 - \nu^2]\, a_n\, x^{n+\sigma} + \sum_{n\ge 0} a_n\, x^{n+\sigma+2} = 0\,. \tag{5.12}$$

Since this must hold for all x, we can now equate to zero the coefficient of each power of x. To do this, in the first sum we make the replacement $n \to n+2$, so that (5.12) is re-expressed as[15]

$$\begin{aligned}\sum_{n\ge 0} \Big\{ [(n+\sigma+2)^2 - \nu^2]\, a_{n+2} + a_n \Big\}\, x^{n+\sigma+2} \\ +(\sigma^2-\nu^2)\, a_0\, x^\sigma + [(\sigma+1)^2 - \nu^2]\, a_1\, x^{\sigma+1} = 0\,. \end{aligned}\tag{5.13}$$

From this we see that

$$a_{n+2} = \frac{a_n}{\nu^2 - (n+\sigma+2)^2}\,, \tag{5.14}$$

for $n \ge 0$. In addition we have, from the two "extra" terms,

$$(\sigma^2-\nu^2)\, a_0 = 0\,, \qquad [(\sigma+1)^2 - \nu^2]\, a_1 = 0\,. \tag{5.15}$$

We begin with the first equation in (5.15). This is the *Indicial Equation.* Notice that we can in general insist, without any loss of generality, that $a_0 \ne 0$. The reason for this is as follows. Suppose a_0 *were* equal to zero. The series (5.11) would then begin with the a_1 term, so it would be a series whose powers of x were $(x^{\sigma+1}, x^{\sigma+2}, x^{\sigma+3}, \ldots)$. But since at the stage when we write (5.11) σ is a completely arbitrary constant, not yet determined, we

[15]Recall that "sending $n \to n+2$ in the first sum" means first setting $n = m+2$, so that the summation over m runs from -2 up to $+\infty$. Then, we write this as the sum from $m = 0$ to $+\infty$ together with the "extra" two terms $m = -2$ and $m = -1$ added on in addition. Finally, we relabel the m summation variable as n.

could as well relabel it by writing $\sigma = \sigma' - 1$. We would then have a series whose powers of x are $(x^{\sigma'}, x^{\sigma'+1}, x^{\sigma'+2}, \ldots)$. But this is exactly what we would have had if the a_0 term were in fact non-zero, after relabelling σ' as σ. So insisting that a_0 be non-zero loses no generality at all.

Proceeding, we then have the indical equation $\sigma^2 - \nu^2 = 0$, i.e.

$$\sigma = \pm\nu\,. \tag{5.16}$$

Now we look at the second equation in (5.15). Since we already know from the indicial equation that $\sigma^2 = \nu^2$, we can rewrite the second equation as

$$(2\sigma + 1)\, a_1 = 0\,. \tag{5.17}$$

Thus either $a_1 = 0$ or else $\sigma = -\frac{1}{2}$. But since we already know from the indicial equation that $\sigma = \pm\nu$, it follows that except in the very special case where $\nu = \frac{1}{2}$, which has to be analysed separately, we must have that $a_1 = 0$. Let us assume that $\nu \ne \pm\frac{1}{2}$, for simplicity. In fact, we shall assume for now that ν takes a generic value, which is not equal to any integer or half integer.

Finally, in the recursion relation (5.14), we substitute the two possible values for σ, i.e. $\sigma = \nu$ or $\sigma = -\nu$. In each of these two cases, the recursion relation then gives us expressions for all the a_n with $n \ge 2$, in terms of a_0 (which is non-zero), and a_1 (which is zero since we are assuming $\nu \ne \pm\frac{1}{2}$).

We can check the radius of convergence of the series solutions, by applying the ratio test. The ratio of successive terms (bearing in mind that $a_1 = 0$, which means all the odd a_n are zero) is given by

$$\frac{a_{n+2}\, x^2}{a_n} = \frac{x^2}{\nu^2 - (n + \sigma + 2)^2}\,, \tag{5.18}$$

where $\sigma = \pm\nu$. In either case, at large n we see that the absolute value of the ratio tends to x^2/n^2, and thus the ratio becomes zero for any fixed x, no matter how large. Thus the radius of convergence is infinite.

To summarise, we have obtained two solutions to the Bessel equation, $y_1(x)$ and $y_2(x)$, given by

$$y_1(x) = x^{\nu} \sum_{n\ge 0} a_n^+ x^n\,, \qquad y_2(x) = x^{-\nu} \sum_{n\ge 0} a_n^- x^n\,, \tag{5.19}$$

where $a_0^\pm$ is arbitrary, $a_1^\pm = 0$, and from the recursion relation (5.14) we have

$$a_{n+2}^+ = \frac{a_n^+}{\nu^2 - (n + \nu + 2)^2}\,, \qquad a_{n+2}^- = \frac{a_n^-}{\nu^2 - (n - \nu + 2)^2}\,. \tag{5.20}$$

It is straightforward to see that these two recursion relations can be solved, to give the two solutions, called $J_\nu(x)$ and $J_{-\nu}(x)$:

$$\boxed{J_\nu(x) = \Big(\frac{x}{2}\Big)^\nu \sum_{p\ge 0} \frac{(-1)^p}{p!\,\Gamma(p+\nu+1)}\Big(\frac{x}{2}\Big)^{2p}\,, \quad J_{-\nu}(x) = \Big(\frac{x}{2}\Big)^{-\nu} \sum_{p\ge 0} \frac{(-1)^p}{p!\,\Gamma(p-\nu+1)}\Big(\frac{x}{2}\Big)^{2p}\,.} \tag{5.21}$$

Here Γ is the Gamma function, which can be defined by

$$\Gamma(z) = \int_0^\infty e^{-t}\, t^{z-1} dt\,. \tag{5.22}$$

If z is an integer, it is related to the factorial function by $\Gamma(n+1) = n!$. In general, it satisfies the relation

$$z\Gamma(z) = \Gamma(z+1)\,. \tag{5.23}$$

The two *Bessel functions* $J_\nu(x)$ and $J_{-\nu}(x)$ are linearly independent, if ν takes a generic real value. This is obvious from the fact that the leading term in $J_\nu(x)$ is proportional to x^ν, while the leading term in $J_{-\nu}(x)$ is proportional to $x^{-\nu}$.

However, if ν is an integer (which, it should be recalled, is generally going to be the case in physical situations where we are solving Laplace's equation for the electrostatic potential), the two Bessel functions become proportional to one another. For example, if $\nu = 1$ we find from (5.21) that

$$J_1(x) = \tfrac12 x - \tfrac1{16}x^3 + \tfrac1{384}x^4 + \cdots\,, \qquad J_{-1}(x) = -\tfrac12 x + \tfrac1{16}x^3 - \tfrac1{384}x^4 + \cdots\,, \tag{5.24}$$

and in fact $J_1(x) = -J_{-1}(x)$. (One might have thought from the expression for $J_{-1}(x)$ in (5.21) that its leading term would be proportional to $1/x$, but the coefficient of the $p = 0$ term in the summation is $1/\Gamma(0)$, and $\Gamma(0)$ is in fact infinite (see (5.23), with $z = 0$, bearing in mind that $\Gamma(1) = 0! = 1$).) It is not hard to see from the series expansions (5.21) that when $\nu = n$ is an integer, we shall have

$$J_{-n}(x) = (-1)^n J_n(x)\,. \tag{5.25}$$

Since when ν is an integer the two generically-independent solutions J_ν and $J_{-\nu}$ to Bessel's equation become linearly dependent, it follows that the actual "second solution" cannot be of the originally-assumed form (5.11) when ν is an integer. In fact, what is missing is that the actual second solution acquires a dependence on $\log x$ when ν is an integer.

The second solution can be constructed by applying a limiting procedure. Essentially, we take the linear combination of $J_\nu(x)$ and $J_{-\nu}(x)$ that vanishes when ν approaches an integer, and divide it by a ν-dependent factor that also vanishes as ν approaches an integer. This ratio of "0/0" is actually finite and non-zero, and provides us with the second solution. Thus, we define

$$Y_\nu(x) = \frac{J_\nu(x)\cos\nu\pi - J_{-\nu}(x)}{\sin\nu\pi}\,. \tag{5.26}$$

We may take $J_\nu(x)$ and $Y_\nu(x)$ to be the two linearly-independent solutions of Bessel's equation (5.10) for arbitrary ν, integer or non-integer. (When ν is not an integer, Y_ν is a nonsingular linear combination of J_ν and $J_{-\nu}$, and so it is linearly independent of J_ν.) The function $J_\nu(x)$ is sometimes called the *Bessel function of the first kind*, and $Y_\nu(x)$ is called the *Bessel function of the second kind*.

It is evident from the series expansion (5.21) that at small x, the Bessel function $J_\nu(x)$ behaves like

$$J_\nu(x) = \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu + \mathcal{O}(x^{\nu+2})\,. \tag{5.27}$$

By means of a more intricate analysis, which requires first finding a suitable integral representation for $J_\nu(x)$, one can show that at large x it takes roughly the form of a cosine function, with a slowly decaying amplitude:

$$J_\nu(x) \sim \sqrt{\frac{2}{\pi x}}\,\cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)\,, \qquad x \longrightarrow \infty\,. \tag{5.28}$$

The $Y_\nu(x)$ Bessel function has a similar decaying oscillatory behaviour at large x:

$$Y_\nu(x) \sim \sqrt{\frac{2}{\pi x}}\,\sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)\,, \qquad x \longrightarrow \infty\,. \tag{5.29}$$

If ν is not an integer its small-x behaviour is just that which is implied by the two series expansions in (5.21), combined according to the definition (5.26). As mentioned previously, $Y_\nu(x)$ involves dependence on $\log x$ if ν is an integer. The small-x behaviour for the first couple of $Y_n(x)$ functions for integer n are:

$$\begin{aligned} Y_0(x) &= \frac{2}{\pi}\Big(\log\tfrac{1}{2}x + \gamma\Big)\Big(1 - \tfrac{1}{4}x^2 + \tfrac{1}{64}x^4 + \cdots\Big) + \frac{1}{2\pi}\Big(x^2 - \tfrac{3}{32}x^4 + \cdots\Big)\,, \\ Y_1(x) &= \frac{1}{\pi}\Big(\log\tfrac{1}{2}x + \gamma\Big)\Big(x - \tfrac{1}{8}x^3 + \tfrac{1}{192}x^5 + \cdots\Big) - \frac{2}{\pi}\Big(\frac{1}{x} + \tfrac{1}{4}x - \tfrac{5}{64}x^3\cdots\Big)\,, \end{aligned} \tag{5.30}$$

where $\gamma \approx 0.5772157$ is the Euler-Mascheroni constant. In general, for $\nu > 0$, the leading-order small-x behaviour of $Y_\nu(x)$ is of the form

$$Y_\nu(x) \sim -\frac{\Gamma(\nu)}{\pi}\left(\frac{2}{x}\right)^\nu\,. \tag{5.31}$$

The three figures below contain plots of the $J_0(x)$, $J_1(x)$ and $J_5(x)$ Bessel functions.

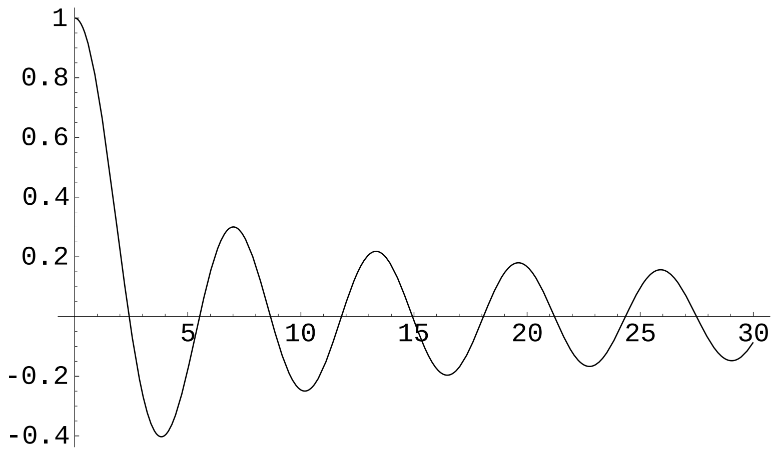

Figure 1: The $J_0(z)$ Bessel Function

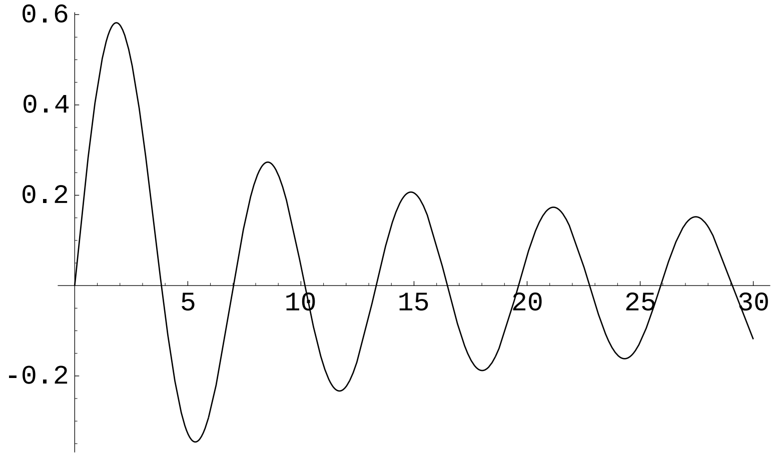

Figure 2: The $J_1(z)$ Bessel Function

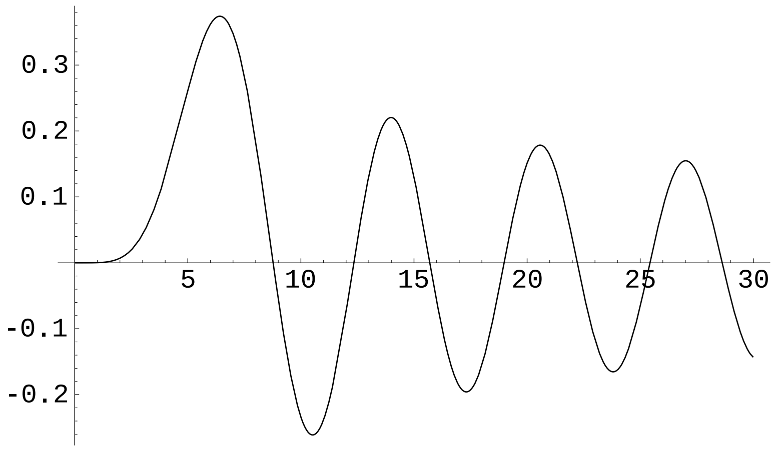

Figure 3: The $J_5(z)$ Bessel Function

5.2 Properties of the Bessel functions

From the asymptotic forms (5.28) and (5.29) of the J_ν and Y_ν Bessel functions, it can be seen to be natural to define also the complex combinations

$$H_\nu^{(1)}(x) = J_\nu(x) + \mathrm{i} Y_\nu(x)\,, \qquad H_\nu^{(2)}(x) = J_\nu(x) - \mathrm{i} Y_\nu(x)\,. \tag{5.32}$$

These are known as the *Hankel functions*, or as "Bessel functions of the third kind." They bear the same relation to J_v and Y_ν as $e^{\pm \mathrm{i}\theta}$ does to $\cos\theta$ and $\sin\theta$.

All the Bessel functions can be shown to satisfy the recurrence relations

$$W_{\nu-1}(x) + W_{\nu+1}(x) = \frac{2\nu}{x}\, W_\nu(x)\,, \tag{5.33}$$

$$W_{\nu-1}(x) - W_{\nu+1}(x) = 2\frac{dW_\nu(x)}{dx}\,, \tag{5.34}$$

where W_ν is taken to be any one of J_ν, Y_ν, $H_\nu^{(1)}$ or $H_\nu^{(2)}$. These relations can be proven directly, using the expression in (5.21) for the series expansion of $J_\nu(x)$.

We have seen from the plots of the J_ν Bessel functions, and from their asymptotic behaviour, that $J_\nu(x)$ has a discrete infinite set of zeros, at points on the x axis that asymptotically approach an equal spacing. Let us say that the m'th zero of $J_\nu(x)$ occurs at

$$x = \alpha_{\nu m}\,, \qquad \text{so} \qquad J_\nu(\alpha_{\nu m}) = 0\,. \tag{5.35}$$

Thus $x = \alpha_{\nu\,1}$ is the location of the first zero, $x = \alpha_{\nu\,2}$ is the location of the second, and so on, as x increases from 0. They occur at definite values of $\alpha_{\nu m}$, though it is not easy to give explicit expressions for $\alpha_{\nu m}$.

Recall that the Bessel equation arose from our separation of variables in cylindrical polar coordinates (ρ, φ, z), and that the independent variable x in the Bessel equation (5.10) was related to the radial coordinate ρ by $x = k\rho$, where k was one of the separation constants in (5.6) and (5.7). We shall now show that the functions $\rho^{1/2}\, J_\nu(\alpha_{\nu n}\rho/a)$, for a fixed $\nu \ge 0$ and with n taking all the positive integer values, form an orthogonal set in the interval $0 \le \rho \le a$.

It follows from Bessel's equation (5.10) that $J_\nu(\alpha_{\nu m}\rho/a)$ satisfies

$$\frac{d}{d\rho}\Big(\rho\, \frac{dJ_\nu(\alpha_{\nu m}\rho/a)}{d\rho}\Big) + \Big(\frac{\alpha_{\nu m}^2 \rho}{a^2} - \frac{\nu^2}{\rho}\Big) J_\nu(\alpha_{\nu m}\rho/a) = 0\,. \tag{5.36}$$

We prove orthogonality by multiplying by $J_\nu(\alpha_{\nu n}\rho/a)$, and then subtracting off the equation where the rôles of m and n are exchanged, to give

$$\begin{aligned} &J_\nu(\alpha_{\nu n}\,\rho/a)\, \frac{d}{d\rho}\Big(\rho\, \frac{d}{d\rho} J_\nu(\alpha_{\nu m}\,\rho/a)\Big) - J_\nu(\alpha_{\nu m}\,\rho/a)\, \frac{d}{d\rho}\Big(\rho\, \frac{d}{d\rho} J_\nu(\alpha_{\nu n}\,\rho/a)\Big) \\ &\quad = \frac{\alpha_{\nu n}^2 - \alpha_{\nu m}^2}{a^2}\, \rho\, J_\nu(\alpha_{\nu m}\,\rho/a)\, J_\nu(\alpha_{\nu n}\,\rho/a)\,. \end{aligned} \tag{5.37}$$

Next, we integrate this from $\rho = 0$ to $\rho = a$. On the left-hand side we integrate by parts, finding that there is now a cancellation of the resulting two integrands, leaving only the "boundary terms." Thus we have

$$\Big[\rho\, J_\nu(\alpha_{\nu n}\,\rho/a)\,\frac{d}{d\rho} J_\nu(\alpha_{\nu m}\,\rho/a)\Big]_0^a - \Big[\rho\, J_\nu(\alpha_{\nu m}\,\rho/a)\,\frac{d}{d\rho} J_\nu(\alpha_{\nu n}\,\rho/a)\Big]_0^a$$
$$= \frac{\alpha_{\nu n}^2 - \alpha_{\nu m}^2}{a^2} \int_0^a J_\nu(\alpha_{\nu m}\,\rho/a)\, J_\nu(\alpha_{\nu n}\,\rho/a)\,\rho\, d\rho\,. \tag{5.38}$$

Recalling that near $\rho = 0$, $J_\nu(\alpha_{\nu n}\,\rho/a)$ is proportional to ρ^ν, we see that with our assumption that $\nu \ge 0$ the lower limits on the left-hand side of (5.38) will give zero. Furthermore, the upper limits will also give zero, since by construction $J_\nu(\alpha_{\nu m}) = 0$. Thus we arrive at the conclusion that for $m \ne n$ (which implies $\alpha_{\nu m} \ne \alpha_{\nu n}$), we shall have

$$\int_0^a J_\nu(\alpha_{\nu m}\,\rho/a)\, J_\nu(\alpha_{\nu n}\,\rho/a)\,\rho\, d\rho = 0\,. \tag{5.39}$$

Having established orthogonality when $m \ne n$, it remains to determine the normalisation of the integral that we get when instead we take $m = n$. To do this, let $x = \alpha_{\nu m}\,\rho/a$, so that

$$\int_0^a [J_\nu(\alpha_{\nu n}\,\rho/a)]^2\,\rho\, d\rho = \frac{a^2}{\alpha_{\nu n}^2} \int_0^{\alpha_{\nu n}} [J_\nu(x)]^2\, x\, dx\,. \tag{5.40}$$

To evaluate the integral on the right-hand side, we integrate by parts, by writing $[J_\nu(x)]^2\, x = \frac12 d/dx(x^2\,[J_\nu(x)]^2) - \frac12 x^2\, d/dx([J_\nu(x)]^2)$, so that

$$\int_{x_1}^{x_2} [J_\nu(x)]^2\, x\, dx = \Big[\tfrac12 x^2\, J_\nu^2\Big]_{x_1}^{x_2} - \int_{x_1}^{x_2} x^2\, J_\nu\, J_\nu'\, dx\,. \tag{5.41}$$

We have also allowed rather more general upper and lower limits of integration x_1 and x_2 here, since then the resulting formula will be of wider applicability. Now use the Bessel equation (5.10) to write $x^2\, J_\nu$ as $\nu^2\, J_\nu - x\, J_\nu' - x^2\, J_\nu''$, so that we get

$$\begin{aligned}
\int_{x_1}^{x_2} [J_\nu(x)]^2\, x\, dx &= \Big[\tfrac12 x^2\, J_\nu^2\Big]_{x_1}^{x_2} - \int_{x_1}^{x_2} \Big(\nu^2\, J_\nu\, J_\nu' - x\, J_\nu'^{\,2} - x^2\, J_\nu'\, J_\nu''\Big)\, dx\,,\\
&= \Big[\tfrac12 x^2\, J_\nu^2\Big]_{x_1}^{x_2} - \int_{x_1}^{x_2} \Big(\tfrac12 \nu^2\, (J_\nu^2)' - \tfrac12 (x^2\, J_\nu'^{\,2})'\Big)\, dx\\
&= \tfrac12 \Big[x^2\, J_\nu^2 - \nu^2\, J_\nu^2 + x^2\, J_\nu'^{\,2}\Big]_{x_1}^{x_2}\,.
\end{aligned} \tag{5.42}$$

In our specific case we have integration limits $x_1 = 0$, $x_2 = \alpha_{\nu n}$. Therefore the first two terms in the final line vanish at both our endpoints (recall that $\alpha_{\nu n}$ are precisely the values of argument for which $J_\nu(\alpha_{\nu n}) = 0$). For the final term, we subtract (5.33) from (5.34) to give

$$J_\nu'(x) = \frac{\nu}{x}\, J_\nu(x) - J_{\nu+1}(x)\,. \tag{5.43}$$

Thus, with our assumption that $\nu \ge 0$ we see that $x^2\, {J'_\nu}^2$ will vanish at $x = 0$. Also, from (5.43) we see that $J'_\nu(\alpha_{\nu n}) = -J_{\nu+1}(\alpha_{\nu n})$, and so

$$\int_0^{\alpha_{\nu n}} [J_\nu(x)]^2\, x\, dx = \tfrac{1}{2}\alpha_{\nu n}^2\, [J_{\nu+1}(\alpha_{\nu n})]^2\,, \tag{5.44}$$

implying finally that

$$\int_0^a J_\nu(\alpha_{\nu m}\, \rho/a)\, J_\nu(\alpha_{\nu n}\, \rho/a)\, \rho\, d\rho = \tfrac{1}{2}a^2\, [J_{\nu+1}(\alpha_{\nu n})]^2\, \delta_{mn}\,. \tag{5.45}$$

We can expand an arbitrary function $f(\rho)$ defined in the interval $0 \le \rho \le a$ in a Fourier-Bessel series

$$\boxed{f(\rho) = \sum_{n\ge 1} A_{\nu n} J_\nu\Big(\alpha_{\nu n}\frac{\rho}{a}\Big)\,.} \tag{5.46}$$

Multiplying by $\rho J_\nu(\alpha_{\nu m}\rho/a)$, integrating over ρ, and using the orthogonality relation (5.45), we can solve for the coefficients $A_{\nu n}$, finding

$$\boxed{A_{\nu n} = \frac{2}{a^2[J_{\nu+1}(\alpha_{\nu n})]^2}\int_0^a d\rho\rho f(\rho) J_\nu\Big(\alpha_{\nu n}\frac{\rho}{a}\Big)\,.} \tag{5.47}$$

Going back to our separation of variables in cylindrical polar coordinates, where ϕ was written in the factorised form $\phi(\rho,\varphi,z) = R(\rho)\Phi(\varphi)Z(z)$, we see that the general solution of Laplace's equation can be written in the form

$$\phi(\rho,\varphi,z) = \sum_m \int dk \Big(A_m(k) J_m(k\rho) + B_m(k) Y_m(k\rho)\Big)\, e^{\mathrm{i}m\varphi}\, e^{kz}\,, \tag{5.48}$$

where $A_m(k)$ and $B_m(k)$ are arbitrary coefficients. (We have assumed here that ϕ is periodic in φ, but that boundary conditions that would restrict k to taking a discrete set of values have not yet been imposed.) In a situation where, for example, the potential was zero on a cylinder at $\rho = a$ (and finite at $\rho = 0$), the continuous integral over k would be replaced by a discrete sum of the form

$$\phi(\rho,\varphi,z) = \sum_{m,n} A_{mn} J_m(\alpha_{mn}\frac{\rho}{a})\, e^{\mathrm{i}m\varphi}\, e^{\alpha_{mn}\, z/a}\,. \tag{5.49}$$

Note that in our separation of variables in cylindrical polar coordinates, there was a separation constant k^2 that arose in the equations (5.6) and (5.7). If this constant had been chosen to be of the opposite sign, then the separated equations would instead have been

$$\frac{d^2 Z}{dz^2} + k^2\, Z = 0\,, \qquad \frac{d^2\Phi}{d\varphi^2} + \nu^2\, \Phi = 0\,, \tag{5.50}$$

$$\frac{d^2 R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} - \Big(k^2 + \frac{\nu^2}{\rho^2}\Big)\, R = 0\,, \tag{5.51}$$

The solutions for Z would then be of the form

$$Z \sim e^{\pm \mathrm{i} k z}\,, \tag{5.52}$$

Writing $x = k\rho$, $y = R$ as before we now obtain the *Modified Bessel Equation*

$$x^2\, y''(x) + x\, y'(x) - (x^2 + \nu^2)\, y = 0\,, \tag{5.53}$$

The two linearly-independent solutions of the modified Bessel equation can be taken to be $I_\nu(x)$ and $K_\nu(x)$, defined by

$$\begin{aligned} I_\nu(x) &\equiv e^{-\frac{1}{2}\pi\nu\mathrm{i}}\, J_\nu(x e^{\frac{1}{2}\pi\mathrm{i}})\,, & (5.54)\\ K_\nu(x) &= \tfrac{1}{2}\pi e^{\frac{1}{2}(\nu+1)\pi\mathrm{i}}\, H_\nu^{(1)}(x e^{\frac{1}{2}\pi\mathrm{i}})\,. & (5.55) \end{aligned}$$

Note that from these definitions we have that when ν is an integer, $\nu = n$,

$$I_{-n}(x) = I_n(x)\,, \qquad K_{-n}(x) = K_n(x)\,. \tag{5.56}$$

The function $I_\nu(x)$ has a simple power-series expansion

$$I_\nu(x) = \sum_{r\ge 0} \frac{1}{r!\,\Gamma(\nu+r+1)}\, \Big(\frac{x}{2}\Big)^{\nu+2r}\,. \tag{5.57}$$

This is convergent for any finite x. At small x we therefore have

$$I_\nu(x) = \frac{1}{\Gamma(\nu+1)}\, \Big(\frac{x}{2}\Big)^{\nu} + \mathcal{O}(x^{\nu+2})\,. \tag{5.58}$$

At large x, it can be shown that $I_\nu(x)$ has the asymptotic form

$$I_\nu(x) \sim \frac{1}{\sqrt{2\pi x}}\, e^x \Big(1 + \mathcal{O}(x^{-1})\Big)\,. \tag{5.59}$$

The $K_\nu(x)$ Bessel functions for the first couple of integer values for ν have the small-x expansions

$$\begin{aligned} K_0(x) &= -(\log \tfrac{1}{2}x + \gamma)(1 + \tfrac{1}{4}x^2 + \tfrac{1}{64}x^4 + \cdots) + \tfrac{1}{4}x^2 + \tfrac{3}{128}x^4 + \cdots\,,\\ K_1(x) &= \tfrac{1}{2}(\log \tfrac{1}{2}x + \gamma)(x + \tfrac{1}{8}x^3 + \tfrac{1}{192}x^5 + \cdots) + \frac{1}{x} - \tfrac{1}{4}x - \tfrac{5}{64}x^3 + \cdots\,. \end{aligned} \tag{5.60}$$

In general, when $\nu > 0$, the leading-order behaviour is

$$K_\nu(x) \sim \tfrac{1}{2}\Gamma(\nu)\, \Big(\frac{2}{x}\Big)^{\nu}\,. \tag{5.61}$$

At large x, $K_\nu(x)$ has the asymptotic form

$$K_\nu(x) \sim \sqrt{\frac{\pi}{2x}}\, e^{-x} \Big(1 + \mathcal{O}(x^{-1})\Big)\,. \tag{5.62}$$

5.3 A boundary-value problem in cylindrical polar coordinates

We can now apply some of the technology of Bessel functions to the solution of electrostatics problems in cylindrical polar coordinates. Consider the following example. A conducting cylinder of height h and radius a is held at zero potential. A flat conductor closes off the cylinder at $z = 0$, and is also at zero potential. The top face, at $z = h$, is held at some specified potential

$$\phi(\rho, \varphi, h) = V(\rho, \varphi)\,. \tag{5.63}$$

The problem is to determine the potential everywhere inside the cavity.

Given that the potential vanishes at $\rho = a$, and it must be finite at $\rho = 0$, it follows that the I_ν and K_ν Bessel functions are not the appropriate ones for this problem. In other words, we must choose the sign of the separation constant k^2 that we had in the original construction (5.6) and (5.7), so that the J_ν and Y_ν Bessel functions appear in the expansion. Specifically, we shall need just the J_ν Bessel functions here, since there will be no logarithmic singularities on the axis of the cylinder.

From (5.6) we see that the z dependence and φ dependence of the separation functions $Z(z)$ and $\Phi(\varphi)$ will be

$$\begin{aligned} Z(z) &\sim \sinh kz\,, \qquad \cosh kz\,, \\ \Phi(\varphi) &\sim \cos \nu\varphi\,, \qquad \sin \nu\varphi\,. \end{aligned} \tag{5.64}$$

The vanishing of the potential on the plate at $z = 0$ means that for $Z(z)$, we shall have only the $\sinh kz$ solution. The periodicity in φ means that ν must be an integer.

The general solution of Laplace's equation for this problem will be

$$\phi(\rho, \varphi, z) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m(\alpha_{mn}\, \rho/a)\,(a_{mn}\, \sin m\varphi + b_{mn}\, \cos m\varphi)\, \sinh(\alpha_{mn}\, z/a)\,. \tag{5.65}$$

The expansion coefficients a_{mn} and b_{mn} are determined by matching this solution to the specified boundary condition (5.63) at $z = h$. Thus we have

$$V(\rho, \varphi) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m(\alpha_{mn}\, \rho/a)\,(a_{mn}\, \sin m\varphi + b_{mn}\, \cos m\varphi)\, \sinh(\alpha_{mn}\, h/a)\,. \tag{5.66}$$

The orthogonality relation (5.45) for the Bessel functions, together with the standard orthogonality for the trigonometric functions, means that all we need to do is to multiply (5.66) by $\rho J_p(\alpha_{pq}\, \rho/a)\, \sin p\varphi$ or $\rho J_p(\alpha_{pq}\, \rho/a)\, \cos p\varphi$ and integrate over ρ and φ in order to read off the integrals that determine the individual coefficients a_{pq} and b_{pq}. It is easy to see that the result is

$$
\begin{aligned}
a_{pq} &= \frac{2}{\pi\, a^2\, \sinh(\alpha_{pq} h/a)\, J_{p+1}(\alpha_{pq})^2} \int_0^{2\pi} d\varphi \int_0^a \rho\, d\rho\, V(\rho,\varphi)\, J_p(\alpha_{pq}\, \rho/a)\, \sin p\varphi\,, \\
b_{pq} &= \frac{2}{\pi\, a^2\, \sinh(\alpha_{pq} h/a)\, J_{p+1}(\alpha_{pq})^2} \int_0^{2\pi} d\phi \int_0^a \rho\, d\rho V(\rho,\varphi)\, J_p(\alpha_{pq}\, \rho/a)\, \cos p\varphi\,.
\end{aligned}
\tag{5.67}
$$

(When $p=0$, we must send $b_{0q} \to \frac{1}{2} b_{0q}$. This is because $\int_0^{2\pi} \cos m\varphi \cos p\varphi d\varphi = \pi \delta_{pq}$ for $p \ne 0$, but it equals 2π if $p=q=0$.)

5.4 Green function in cylindrical polar coordinates

Recall that in section 4.10 we obtained an expansion for $|\vec{r}-\vec{r}\,'|^{-1}$ (i.e. the Green function in free space) in spherical polar coordinates, in terms of the spherical harmonics. From this, we were then able to obtain an expansion for the Green function for the Dirichlet boundary-value problem on a sphere.

The essential idea involved in obtaining the expansion for $|\vec{r}-\vec{r}\,'|^{-1}$ in terms of spherical harmonics was to consider the most general solution of Laplace's equation in each of the regions $r<r'$ and $r>r'$, with the radial functions $r^{-1}R(r)$ in each of the two regions being taken to be the ones appropriate to the regularity requirements in that region. Thus, with the radial functions being r^ℓ and $r^{-(\ell+1)}$, we took the former choice for the region $r<r'$, and the latter for the region $r>r'$. The discontinuity between the two solutions of Laplace's equation as the radial coordinate made the transition from $r<r'$ to $r>r'$ was responsible for producing the delta function factor $\delta(r-r')$ when the Laplacian of the expressions was carefully evaluated. The $\delta(\theta-\theta')$ and $\delta(\varphi-\varphi')$ factors, on the other hand, arose through the completeness relation (4.140).

The same idea can be applied to obtain an expansion for $|\vec{r}-\vec{r}\,'|^{-1}$ in cylindrical polar coordinates. This time, the radial functions $R(\rho)$ are either $J_\nu(k\rho)$ and $Y_\nu(k\rho)$ or else $I_\nu(k\rho)$ and $K_\nu(k\rho)$, depending upon whether the separation constant k^2 enters the equations as in (5.6) and (5.7) or else in (5.50) and (5.51). We can make the (5.50) and (5.51) sign choice here, and so the radial functions are $I_\nu(k\rho)$ and $K_\nu(k\rho)$, and the functions $Z(z)$ and $\Phi(\varphi)$ are of the forms

$$
\begin{aligned}
Z(z) &\sim \cos kz \quad \text{or} \quad \sin kz\,, \\
\Phi(\varphi) &\sim \cos \nu\varphi \quad \text{or} \quad \sin \nu\varphi\,.
\end{aligned}
\tag{5.68}
$$

Single-valuedness of $\Phi(\varphi)$ under $\varphi \to \varphi + 2\pi$ implies that we must take $\nu = m = $integer.

The relevant completeness relations that will generate the delta function factors $\delta(z-z')$ and $\delta(\varphi-\varphi')$ are the standard ones from Fourier analysis:

$$\begin{aligned}
\delta(z-z') &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dk e^{\mathrm{i}k(z-z')} = \frac{1}{\pi}\int_0^{\infty} dk\cos k(z-z')\,, \\
\delta(\varphi-\varphi') &= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{\mathrm{i}m(\varphi-\varphi')}\,.
\end{aligned} \tag{5.69}$$

The Green function can now be expanded in the form

$$G(\vec{r},\vec{r}\,') = \frac{1}{2\pi^2}\sum_{m=-\infty}^{\infty}\int_0^{\infty} dk\, R_{(k,m)}(\rho,\rho')\, e^{\mathrm{i}m(\varphi-\varphi')}\,\cos k(z-z')\,, \tag{5.70}$$

where $R_{(k,m)}(\rho,\rho')$ is the appropriate radial function of ρ, which in general must be some linear combination of $I_m(k\rho)$ and $K_m(k\rho)$. It will take two different forms in the two regions $\rho<\rho'$ and $\rho>\rho'$. From our construction of the general solution of Laplace's equation in cylindrical polar coordinates, it is manifest that $G(\vec{r},\vec{r}\,')$ in (5.70) satisfies Laplace's equation everywhere except at the transition where $\rho=\rho'$.

From the asymptotic forms of $I_\nu(x)$ and $K_\nu(x)$ described in section 5.2, it is evident that the solution well behaved at small ρ must be proportional to I_ν, whilst the solution well behaved at large ρ must be proportional to K_ν. Because of the symmetry of the Green function under exchanging ρ and ρ', we can expect that $R_{(k,m)}(\rho,\rho')$ will be proportional to $I_m(k\rho)K_m(k\rho')$ when $\rho<\rho'$, and proportional to $K_m(\rho)I_m(\rho')$ when $\rho>\rho'$. (We saw precisely the analogous feature previously, in the expansion in spherical polar coordinates.)

Since we still need to determine the constants of proportionality in each case, let us for now write

$$R_{(k,m)}(\rho,\rho') = \psi_1(k\rho)\psi_2(k\rho')\vartheta(\rho'-\rho) + \psi_2(k\rho)\psi_1(k\rho')\vartheta(\rho-\rho')\,, \tag{5.71}$$

where $\psi_1(x)$ is some constant multiple of $I_m(x)$, and $\psi_2(x)$ is some constant multiple of $K_m(x)$. Differentiating, and making use of the standard properties of the derivative of the Heaviside ϑ function, and the fact that $\delta(\rho-\rho')f(\rho')=\delta(\rho-\rho')f(\rho)$, we see that

$$\begin{aligned}
\frac{dR_{(k,m)}(\rho,\rho')}{d\rho} &= \psi_2(k\rho')\frac{d\psi_1(k\rho)}{d\rho}\theta(\rho'-\rho) + \psi_1(k\rho')\frac{d\psi_2(k\rho)}{d\rho}\theta(\rho-\rho')\,, \\
\frac{d^2R_{(k,m)}(\rho,\rho')}{d\rho^2} &= \psi_2(k\rho')\frac{d^2\psi_1(k\rho)}{d\rho^2}\theta(\rho'-\rho) + \psi_1(k\rho')\frac{d^2\psi_2(k\rho)}{d\rho^2}\theta(\rho-\rho') \\
&\quad + \Big[\psi_1(k\rho)\frac{d\psi_2(k\rho)}{d\rho} - \psi_2(k\rho)\frac{d\psi_1(k\rho)}{d\rho}\Big]\,\delta(\rho-\rho')\,.
\end{aligned} \tag{5.72}$$

When we apply the Laplace operator to the expression (5.70), all the terms in (5.72) where the ϑ functions remain will combine with the z and φ derivative terms to give the "bulk" result that $G(\vec{r},\vec{r}\,')$ satisfies Laplace's equation, in the way we discussed above. The only terms left over will be those in (5.72) involving the delta function. For these, we need to derive a simple result about properties of the solutions of the modified Bessel equation (5.53). Suppose $\psi_1(x)$ and $\psi_2(x)$ are two such solutions. Plugging ψ_1 into (5.53) and multiplying by ψ_2, and then subtracting the expession where the roles of ψ_1 and ψ_2 are exchanged, we obtain

$$x\psi_2(x\psi_1')' - x\psi_1(x\psi_2')' = 0\,. \tag{5.73}$$

(Note that $x^2y'' + xy'$ can be written as $x(xy')'$.) Dividing out by x, we can then rewrite this as

$$(x\psi_2\psi_1')' - x\psi_2'\psi_1' - (x\psi_1\psi_2')' + x\psi_1'\psi_2' = 0\,, \tag{5.74}$$

and hence $[x(\psi_1'\psi_2 - \psi_2'\psi_1)]' = 0$. From this, we deduce that for any two solutions of the modified Bessel equation, their Wronskian $W(\psi_1,\psi_2) \equiv \psi_1\psi_2' - \psi_2\psi_1'$ satisfies

$$\psi_1\psi_2' - \psi_2\psi_1' = \frac{c}{x}\,, \tag{5.75}$$

where c is some constant that depends on the choice of ψ_1 and ψ_2.

We can calculate the Wronskian for $I_\nu(x)$ and $K_\nu(x)$ easily, by using the small-x expansions (5.58) and (5.61). This leads to the conclusion that

$$I_\nu(x)K_\nu'(x) - K_\nu(x)I_\nu'(x) = -\frac{1}{x}\,. \tag{5.76}$$

Going back to our trial expansion (5.70), we see that we should choose

$$R_{(k,m)}(\rho,\rho') = 4\pi I_m(k\rho)K_m(k\rho')\vartheta(\rho'-\rho) + 4\pi K_m(k\rho)I_m(k\rho')\vartheta(\rho-\rho')\,. \tag{5.77}$$

This then ensures that $G(\vec{r},\vec{r}\,')$ in (5.70) satisfies

$$\nabla^2 G(\vec{r},\vec{r}\,') = -4\pi\delta^3(\vec{r}-\vec{r}\,') = -\frac{4\pi}{\rho}\,\delta(\rho-\rho')\delta(\varphi-\varphi')\delta(z-z')\,. \tag{5.78}$$

To summarise, we have shown that in cylindrical polar coordinates we have the expansion

$$\boxed{\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{2}{\pi}\sum_{m=-\infty}^{\infty}\int_0^\infty dk\, I_m(k\rho_<)K_m(k\rho_>)e^{\mathrm{i}m(\varphi-\varphi')}\,\cos k(z-z')\,,} \tag{5.79}$$

where the notation here is that $\rho_<$ means whichever of ρ and ρ' is the smaller, and $\rho_>$ means whichever of them is the larger.

Recall that for the integer degrees, as we have here, I_n and K_n are each the same for positive and negative n (see eqn (5.56)). Thus we may also write (5.79) as

$$\boxed{\begin{aligned}\frac{1}{|\vec{r}-\vec{r}\,'|} &= \frac{4}{\pi}\int_0^\infty dk\,\Big(\tfrac{1}{2}I_0(k\rho_<)K_0(k\rho_>) + \sum_{m\ge 1} I_m(k\rho_<)K_m(k\rho_>)\cos m(\varphi-\varphi')\Big)\\ &\qquad \times \cos k(z-z')\,, \qquad (5.80)\end{aligned}}$$

6 Multipole Expansion

The multipole expansion provides a way of organising the expression for the electrostatic potential due to a localised distribution of charges, as a sum over terms proportional to the total charge, the dipole moment, the quadrupole moment, and so on. In order to discuss this, it will be convenient first to introduce an index notation for vectors and tensors in three-dimensional Cartesian space.

6.1 Index notation for Cartesian vectors and tensors

Until now, we have typically been writing a vector V as an ordered triplet of components, in the form

$$\vec{V} = (V_x, V_y, V_z)\,. \tag{6.1}$$

In the index notation, we instead label the three components by the numbers 1, 2 and 3, rather than by x, y and z. Thus we write

$$\vec{V} = (V_1, V_2, V_3)\,. \tag{6.2}$$

In the same way, instead of writing $\vec{r} = (x, y, z)$, we can write it as

$$\vec{r} = (x_1, x_2, x_3)\,. \tag{6.3}$$

In other words, we call the three Cartesian coordinates x_1, x_2 and x_3, rather than x, y and z.

The scalar product $\vec{A}\cdot\vec{B}$ between any two vectors $\vec{A}$ and $\vec{B}$ can now be written as

$$\vec{A}\cdot\vec{B} = \sum_{i=1}^{3} A_i B_i\,. \tag{6.4}$$

At this point, the index notation is looking somewhat clumsy and long-winded. It becomes much simpler to write if we now adopt the *Einstein summation convention.* The idea here is that in any valid vector or tensor expression written using index notation, if a particular index occurs *exactly twice* in a term, then it will *always* be summed over, as in the right-hand side in (6.4). It is therefore redundant to write the summation explicitly, and so we can simply rewrite (6.4) as

$$\vec{A}\cdot\vec{B} = A_i B_i\,. \tag{6.5}$$

By the Einstein summation convention, it is understood that the index i is to be summed over.

The i index in (6.5) is called a *dummy index*, meaning that it is just an index "internal" to the term $A_i B_i$ that is summed over. The term could just as well be written as $A_j B_j$ or $A_k B_k$, etc. The dummy index is like a summation index in a computer programme; for example, in *Mathematica* one would write $A_i B_i$ as

$$\mathrm{Sum}[A[i]B[i], \{i, 1, 3\}]\,, \tag{6.6}$$

and obviously any other name could equally well have been chosen for the summation variable.

The index notation is extremely useful, but there are pitfalls for the unwary. Suppose, for example, we wish to write $(\vec{A}\cdot\vec{B})(\vec{C}\cdot\vec{D})$ in index notation. It will be

$$(\vec{A}\cdot\vec{B})(\vec{C}\cdot\vec{D}) = A_i B_i C_j D_j\,. \tag{6.7}$$

It is *absolutely essential* to make sure that a different index is chosen for the second factor, $\vec{C}\cdot\vec{D}$, that has not been used in writing the first factor, $\vec{A}\cdot\vec{B}$. Thus, for example,

$$A_i B_i C_i D_i \tag{6.8}$$

is completely meaningless. There will *never* be any occasion when a valid vector or tensor expression has a term where the same index occurs more than twice. It should be clear that (6.8) is meaningless and that it is not what is wanted.

In fact in a valid vector or tensor expression, a particular index can appear either once, or twice (or not at all) in each term. For example, suppose we want to write $\vec{V} = (\vec{A}\cdot\vec{B})\vec{C} + (\vec{E}\cdot\vec{F})\vec{G}$ in index notation. This is a vector-valued expression, and so we need a *free index* that labels the three components. We can write it as

$$V_i = A_j B_j C_i + E_j F_j G_i\,. \tag{6.9}$$

This could equally well be written, for example, as

$$V_i = A_j B_j C_i + E_k F_k G_i\,, \tag{6.10}$$

since the summation over the dummy index j in the first term on the right-hand side of (6.9) is completely independent of the summation over the dummy index j in the second term. But we must *not*, under any circumstances, choose the label i for either of the dummy indices in (6.10), since i is already being used as the free index.

The reason for emphasising so strongly the need to be careful about not violating the rules when using indices is that it is by far the most common mistake that people make when they first meet the index notation. If you find you have written a term such as $A_i B_i C_i$, or $A_i B_i C_i D_i$, it is simply wrong; there is no point in continuing the calculation until the source of the trouble has been located and corrected.

One final piece of notation before we move on concerns the gradient operator $\vec{\nabla} = (\nabla_1, \nabla_2, \nabla_3) = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$. In index notation we therefore have

$$\nabla_i = \frac{\partial}{\partial x_i}\,. \tag{6.11}$$

It will be convenient to abbreviate the writing of $\partial/\partial x_i$, by defining

$$\partial_i = \frac{\partial}{\partial x_i}\,. \tag{6.12}$$

Note that by the rules of partial differentiation, we have

$$\partial_i x_j = \delta_{ij}\,. \tag{6.13}$$

Note also, since it is a point that is often overlooked by those meeting the index notation for the first time, that

$$\delta_{ii} = 3\,, \tag{6.14}$$

and so also we have $\partial_i\, x_i = 3$. Another useful observation is that

$$\delta_{ij} A_j = A_i \tag{6.15}$$

for any vector $\vec{A}$.

6.2 Multipole expansion in Cartesian coordinates

Now back to the multipole expansion. Consider the electrostatic potential of N point charges q_a, located at fixed positions $\vec{r}_a$. It is given by

$$\phi(\vec{r}) = \sum_{a=1}^{N} \frac{q_a}{|\vec{r} - \vec{r}_a|}\,. \tag{6.16}$$

In the continuum limit, the potential due to a charge distrubution characterised by the charge density $\rho(\vec{r})$ is given by

$$\phi(\vec{r}) = \int \frac{\rho(\vec{r}\,')d^3\vec{r}\,'}{|\vec{r}-\vec{r}\,'|}\,. \tag{6.17}$$

Since we shall assume that the charges are confined to a finite region, it is useful to perform a *multipole expansion* of the potential far from the region where the charges are located. This amounts to an expansion in inverse powers of $r = |\vec{r}|$. This can be achieved by performing a Taylor expansion of $|\vec{r}-\vec{r}\,'|^{-1}$.

Recall that in one dimension, Taylor's theorem gives

$$f(x+a) = f(x) + af'(x) + \frac{a^2}{2!}f''(x) + \frac{a^3}{3!}f'''(x) + \cdots\,. \tag{6.18}$$

In three dimensions, the analogous expansion is

$$f(\vec{r}+\vec{a}) = f(\vec{r}) + a_i\partial_i f(\vec{r}) + \frac{1}{2!}a_ia_j\partial_i\partial_j f(\vec{r}) + \frac{1}{3!}a_ia_ja_k\partial_i\partial_j\partial_k f(\vec{r}) + \cdots\,. \tag{6.19}$$

We now apply this 3-dimensional Taylor expansion to the function $f(\vec{r}) = 1/|\vec{r}| = 1/r$, taking $\vec{a} = -\vec{r}\,'$. This gives

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{r} - x_i'\partial_i\frac{1}{r} + \frac{1}{2!}x_i'x_j'\partial_i\partial_j\frac{1}{r} - \frac{1}{3!}x_i'x_j'x_k'\partial_i\partial_j\partial_k\frac{1}{r} + \cdots\,. \tag{6.20}$$

Now since $r^2 = x_jx_j$, it follows that $\partial_i r^2 = 2r\,\partial_i r = 2x_i$, and so

$$\partial_i r = \frac{x_i}{r}\,. \tag{6.21}$$

Note that we have (assuming $r \neq 0$) that

$$\partial_i\partial_i\frac{1}{r} = \partial_i\Big(-\frac{x_i}{r^3}\Big) = -\frac{3}{r^3} + \frac{3x_i}{r^4}\,\frac{x_i}{r} = 0\,, \tag{6.22}$$

or, in other words

$$\nabla^2\frac{1}{r} = 0\,. \tag{6.23}$$

A consequence of this is that the multiple derivatives

$$\partial_i\partial_j\frac{1}{r}\,,\qquad \partial_i\partial_j\partial_k\frac{1}{r}\,,\qquad \partial_i\partial_j\partial_k\partial_\ell\frac{1}{r}\,,\qquad \cdots \tag{6.24}$$

are all *traceless* on any pair of indices:

$$\delta_{ij}\partial_i\partial_j\frac{1}{r} = 0\,,\qquad \delta_{ij}\partial_i\partial_j\partial_k\frac{1}{r} = 0\,,\qquad \text{etc.} \tag{6.25}$$

We can use this property in order to replace the quantities

$$x_i'x_j'\,,\qquad x_i'x_j'x_k'\,,\qquad \cdots \tag{6.26}$$

that multiply the derivative terms in (6.20) by the totally tracefree quantities

$$(x'_i x'_j - \tfrac{1}{3}\delta_{ij} r'^2)\,, \qquad (x'_i x'_j x'_k - \tfrac{1}{5}[x'_i \delta_{jk} + x'_j \delta_{ik} + x'_k \delta_{ij}] r'^2)\,, \qquad \cdots \tag{6.27}$$

where $r'^2 = x'_i x'_i$. (We can do this because the trace terms that we are subtracting out here give zero when they are contracted onto the multiple derivatives of $1/r$ in (6.20).) It therefore follows from (6.17) and (6.20) that we have

$$\begin{aligned}\phi(\vec r) &= \frac{1}{r}\int \rho(\vec r\,')d^3\vec r\,' - \Big(\partial_i \frac{1}{r}\Big)\int x'_i \rho(\vec r\,')d^3\vec r\,' + \tfrac{1}{2}\Big(\partial_i\partial_j \frac{1}{r}\Big)\int (x'_i x'_j - \tfrac{1}{3}\delta_{ij} r'^2)\rho(\vec r\,')d^3\vec r\,' \\ &\quad -\tfrac{1}{6}\Big(\partial_i\partial_j\partial_k \frac{1}{r}\Big)\int (x'_i x'_j x'_k - \tfrac{1}{5}[x'_i\delta_{jk} + x'_j\delta_{ik} + x'_k\delta_{ij}]r'^2)\rho(\vec r\,')d^3\vec r\,' + \cdots\,. \end{aligned} \tag{6.28}$$

The expansion here can be written as

$$\phi(\vec r) = \frac{Q}{r} - p_i \partial_i \frac{1}{r} + \frac{1}{3\times 2!} Q_{ij}\partial_i\partial_j \frac{1}{r} - \frac{1}{5\times 3!} Q_{ijk}\partial_i\partial_j\partial_k \frac{1}{r} + \cdots \tag{6.29}$$

where

$$\begin{aligned} Q &= \int \rho(\vec r\,')d^3\vec r\,'\,, \\ p_i &= \int x'_i \rho(\vec r\,')d^3\vec r\,'\,, \\ Q_{ij} &= \int (3x'_i x'_j - \delta_{ij} r'^2)\rho(\vec r\,')d^3\vec r\,'\,, \\ Q_{ijk} &= \int (5x'_i x'_j x'_k - [x'_i\delta_{jk} + x'_j\delta_{ik} + x'_k\delta_{ij}]r'^2)\rho(\vec r\,')d^3\vec r\,'\,, \end{aligned} \tag{6.30}$$

and so on. The quantity Q is the total charge of the system, p_i is the dipole moment, Q_{ij} is the quadrupole moment, and Q_{ijk}, $Q_{ijk\ell}$, etc., are the higher multipole moments. Note that by construction, all the multipole moments with two or more indices are symmetric and traceless on all indices.

Note that the terms in the multipole expansion (6.29) do indeed fall off with increasing inverse powers of r. For example, the dipole term is given by

$$\phi_{\rm Dipole} = -p_i \partial_i \frac{1}{r} = \frac{p_i x_i}{r^3} = \frac{p_i n_i}{r^2}\,, \tag{6.31}$$

which falls off like $1/r^2$, since $n_i \equiv x_i/r$ is a unit-length vector. The quadrupole term is given by

$$\phi_{\rm Quadrupole} = \tfrac{1}{6} Q_{ij}\partial_i\partial_j \frac{1}{r} = \tfrac{1}{6} Q_{ij} \frac{(3x_i x_j - r^2\delta_{ij})}{r^5} = \tfrac{1}{2} Q_{ij} \frac{x_i x_j}{r^5} = \tfrac{1}{2} Q_{ij} \frac{n_i n_j}{r^3}\,, \tag{6.32}$$

which falls off like $1/r^3$. (The penultimate equality above follows because Q_{ij} is traceless.)

In summary, we see that the multipole expansion of the potential due to a localised charge distribution takes the form

$$\boxed{\phi(\vec{r}) = \frac{Q}{r} + \frac{\vec{p}\cdot\vec{n}}{r^2} + \frac{1}{2!}\frac{Q_{ij}n_i n_j}{r^3} + \frac{1}{3!}\frac{Q_{ijk}n_i n_j n_k}{r^4} + \cdots\,. \qquad (6.33)}$$

The electric field due to the monopole potential $\phi_{\rm Monopole} = Q/r$ is the familiar one

$$\vec{E}_{\rm Monopole} = \frac{Q\vec{r}}{r^3} = \frac{Q\vec{n}}{r^2}\,, \tag{6.34}$$

which falls off as the square of the distance. For the dipole potential (6.31), the electric field is easily calculated using index notation:

$$\partial_i\Big(\frac{p_j x_j}{r^3}\Big) = \frac{p_j\delta_{ij}}{r^3} - \frac{3p_j x_j x_i}{r^5} = -\frac{3n_i n_j p_j - p_i}{r^3}\,, \tag{6.35}$$

and hence

$$\boxed{\vec{E}_{\rm Dipole} = \frac{3\vec{n}\,(\vec{n}\cdot\vec{p}) - \vec{p}}{r^3}\,. \qquad (6.36)}$$

This falls off as the cube of the distance. The electric fields for the higher multipole terms can be calculated in a similar way.

The total charge Q (the electric monopole moment) is of course a single quantity. The dipole moment p_i is a 3-vector, so it has three independent components in general. The quadrupole moment Q_{ij} is a symmetric 2-index tensor in three dimensions, which would mean $3\times 4/2 = 6$ independent components. But it is also traceless, $Q_{ii}=0$, which is one condition. Thus there are $6-1=5$ independent components.

The octopole moment Q_{ijk} is a 3-index symmetric tensor, which would mean $3\times 4\times 5/3! = 10$ independent components. But it is also traceless, $Q_{iij}=0$, which is 3 conditions. Thus the octopole has in general $10-3=7$ independent components. It is straightforward to see in the same way that the 2^ℓ-pole moment

$$Q_{i_1 i_2\cdots i_\ell} = (2\ell-1)\int (x'_{i_1}x'_{i_2}\cdots x'_{i_\ell} - \hbox{traces})\rho(\vec{r}\,')d^3\vec{r}\,'\,, \qquad \ell\ge 1\,, \tag{6.37}$$

has $(2\ell+1)$ independent components.

6.3 Multipole expansion using spherical harmonics

In fact, the multipole expansion (6.29) is equivalent to an expansion in spherical polar coordinates, using the spherical harmonics $Y_{\ell m}(\theta,\phi)$:

$$\phi(r,\theta,\phi) = \sum_{\ell=0}^\infty\sum_{m=-\ell}^{\ell} B_{\ell m}\,Y_{\ell m}(\theta,\phi)\,\frac{1}{r^{\ell+1}}\,. \tag{6.38}$$

At a given value of ℓ the terms fall off like $r^{-\ell-1}$, and there are $(2\ell+1)$ of them, with coefficients $B_{\ell m}$, since m ranges over the integers $-\ell \le m \le \ell$. For each value of ℓ, there is a linear relationship between the $(2\ell+1)$ components of $B_{\ell m}$ and the $(2\ell+1)$ components of the multipole moments Q, p_i. Q_{ij}, Q_{ijk}, etc. Likewise, for each ℓ there is a linear relationship between $r^{-\ell-1}\, Y_{\ell m}(\theta,\varphi)$ and the set of functions $\partial_{i_1}\partial_{i_2}\cdots\partial_{i_\ell} r^{-1}$.

Consider, for example, $\ell=1$. The three functions $Z_i \equiv \partial_i r^{-1} = -x_i/r^3$ are given by

$$Z_1 = -\frac{\sin\theta\cos\varphi}{r^2}\,,\qquad Z_2 = -\frac{\sin\theta\sin\varphi}{r^2}\,,\qquad Z_3 = -\frac{\cos\theta}{r^2}\,,\tag{6.39}$$

when expressed in terms of spherical polar coordinates, for which

$$x = r\sin\theta\cos\varphi\,,\qquad y = r\sin\theta\sin\varphi\,,\qquad z = r\cos\theta\,.\tag{6.40}$$

On the other hand, the $\ell=1$ spherical harmonics are given by (see (4.108))

$$Y_{11} = -\sqrt{\frac{3}{8\pi}}\,\sin\theta\, e^{\mathrm{i}\varphi}\,,\qquad Y_{10} = \sqrt{\frac{3}{4\pi}}\,\cos\theta\,,\qquad Y_{1,-1} = \sqrt{\frac{3}{8\pi}}\,\sin\theta\, e^{-\mathrm{i}\varphi}\,.\tag{6.41}$$

Thus we see that

$$Z_1 = \sqrt{\frac{8\pi}{3}}\,\frac{(Y_{11}-Y_{1,-1})}{2r^2}\,,\qquad Z_2 = \sqrt{\frac{8\pi}{3}}\,\frac{(Y_{11}+Y_{1,-1})}{2\mathrm{i}\, r^2}\,,\qquad Z_3 = -\sqrt{\frac{4\pi}{3}}\,\frac{Y_{10}}{r^2}\,.\tag{6.42}$$

Analogous relations can be seen for all higher values of ℓ.

Working directly with the spherical harmonics, we may substitute the expansion (4.134) for $|\vec r - \vec r\,'|^{-1}$ into (6.17), obtaining

$$\phi(\vec r) = 4\pi \sum_{\ell,m} \frac{q_{\ell m}}{(2\ell+1)}\,\frac{1}{r^{\ell+1}}\, Y_{\ell m}(\theta,\varphi)\,,\tag{6.43}$$

where the multipole moments $q_{\ell m}$ are given by

$$q_{\ell m} \equiv \int \rho(\vec r\,')\, r'^{\ell}\, \bar Y_{\ell m}(\theta',\varphi') d^3\vec r\,'\,.\tag{6.44}$$

Note that because of (4.103), they satisfy

$$\bar q_{\ell m} = (-1)^m\, q_{\ell,-m}\,.\tag{6.45}$$

Clearly, in view of (6.45), the total number of real quantities encoded in $q_{\ell m}$ for a given value of ℓ is $2\ell+1$, which is exactly the same as the number of independent components $Q_{i_1\cdots i_\ell}$ in the ℓ'th multipole moment tensor. Using the expressions (4.111) for the first few spherical harmonics, we can see that the $q_{\ell m}$ for $\ell=0$, 1 and 2 are related to Q, p_i and Q_{ij}

by

$$
\begin{aligned}
q_{00} &= \frac{1}{\sqrt{4\pi}}\, Q\,, \\
q_{11} &= -\sqrt{\frac{3}{8\pi}}\,(p_1 - \mathrm{i} p_2)\,, \qquad q_{10} = \sqrt{\frac{3}{4\pi}}\, p_3\,, \\
q_{22} &= \tfrac{1}{12}\sqrt{\frac{15}{2\pi}}\,(Q_{11} - 2\mathrm{i} Q_{12} - Q_{22})\,, \quad q_{21} = -\tfrac{1}{3}\sqrt{\frac{15}{8\pi}}\,(Q_{13} - \mathrm{i} Q_{23})\,, \quad q_{20} = \tfrac{1}{2}\sqrt{\frac{5}{4\pi}}\, Q_{33}\,.
\end{aligned}
\tag{6.46}
$$

(The expressions for $q_{\ell m}$ with negative m follow from (6.45).) Analogous relations hold for all the $q_{\ell m}$.

6.4 Another construction of the spherical harmonics

The multipole expansion using Cartesian coordinates, which we discussed in section 6.2, points the way to a very simple construction of the spherical harmonics that sidesteps the need for the elaborate investigation of the separation of variables and the properties of the associated Legendre functions that we examined in great depth previously. We saw in section 6.2 that the tensor

$$\partial_{i_1} \cdots \partial_{i_\ell} \frac{1}{r} \tag{6.47}$$

satisfies Laplace's equation, is symmetric under the exchange of any pair of indices, and it is traceless with respect to the contraction of any pair of indices. Let us define

$$\psi_\ell \equiv C_{i_1 \cdots i_\ell}\, \partial_{i_1} \cdots \partial_{i_\ell} \frac{1}{r}\,, \tag{6.48}$$

where $C_{i_1 \cdots i_\ell}$ is a constant, traceless, symmetric tensor.

In view of the fact that $\partial_i r = x_i/r = n_i$, and bearing in mind that

$$n_1 = \sin\theta\cos\varphi\,, \qquad n_2 = \sin\theta\sin\varphi\,, \qquad n_3 = \cos\theta\,, \tag{6.49}$$

it is evident that when we calculate the result of acting with all the derivatives in (6.48), we shall find that ψ_ℓ is of the form

$$\psi_{(\ell)} = \frac{1}{r^{\ell+1}}\, Y(\theta,\varphi)\,, \tag{6.50}$$

where $Y(\theta,\varphi)$ is some function that depends on θ and φ *but not on* r.[16]

[16]One way to see this is to note that r has the dimensions of length, i.e. $[r] = L$, whilst ∂_i has dimension L^{-1}. Therefore we have that $[\psi_{(\ell)}] = L^{-\ell-1}$. Since $\psi_{(\ell)}$ can be written in terms of r, θ and φ, and since θ and φ are dimensionless, it must be that $\psi_{(\ell)}$ is of the form $r^{-\ell-1}$ multiplied by a (necessarily dimensionless) function of θ and φ.

Since we have seen that $\nabla^2\psi_{(\ell)} = 0$, and since we have

$$\frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2\frac{\partial\phi}{\partial r}\Big) + \frac{1}{r^2}\nabla^2_{(\theta,\varphi)}\,\phi = 0\,, \tag{6.51}$$

it follows that $Y(\theta,\varphi)$, which is defined by (6.48) and (6.50), must satisfy

$$\nabla^2_{(\theta,\varphi)}\,Y(\theta,\varphi) = -\ell(\ell+1)Y(\theta,\varphi)\,. \tag{6.52}$$

This is precisely the equation satisfied by the spherical harmonics $Y_{\ell m}(\theta,\varphi)$. So we have shown that the construction (6.48), together with (6.50), gives spherical harmonics.

It remains to verify that we get the full set of spherical harmonics by this means. In fact, we essentially already did this in section (6.2). We saw there that there are $(2\ell+1)$ independent components to an ℓ-index symmetric traceless tensor. Thus, the construction in (6.48), together with (6.50), yields all $(2\ell+1)$ spherical harmonics at each level ℓ. We obtain them all by considering all possible symmetric traceless tensors $C_{i_1\cdots i_\ell}$.

The relation between this construction and the usual one can be seen for the first few values of ℓ by comparing (4.108) and (4.111).

6.5 Multipole expansion of the energy in an external field

Recall that the energy U of a localised charge distribution was discussed in section 1.12, and shown to be given by (1.105)

$$U_{\rm int} = \tfrac12\int \rho(\vec r)\phi(\vec r)d^3\vec r. \tag{6.53}$$

We have added the subscript "int" here to emphasise that that result gave the "self energy" or "internal energy" of the charge distribution itself, in its own self-generated electrostatic field.

A different question, with a different answer, concerns the energy of a charge distribution in an *externally-applied* electrostatic field. If the external field is expressed in terms of the potential $\Phi(\vec r)$ (as opposed to the potential $\phi(\vec r)$ in (6.53), which is the potential due to the charge distribution itself), then the "external" energy of the system is simply calculated by integrating up the energy of assembling all the charges that form the distribution $\rho(\vec r)$. This gives

$$U_{\rm ext} = \int \rho(\vec r)\Phi(\vec r)d^3\vec r\,. \tag{6.54}$$

We shall assume that the external electric field $\vec E = -\vec\nabla\Phi$ is generated by distant sources (i.e. distant charges), so that we can take $\nabla^2\Phi = 0$ in the region where the localised charge distribution ρ is non-zero. Let us choose the origin to lie in the vicinity of the localised charge distribution, and furthermore we assume that the external field is a *slowly varying* function of $\vec{r}$ in this region. We may then Taylor expand $\Phi(\vec{r})$, to give

$$\Phi(\vec{r}) = \Phi(0) + x_i \partial_i \Phi(0) + \tfrac{1}{2} x_i x_j \partial_i \partial_j \Phi(0) + \cdots . \tag{6.55}$$

(Of course $\partial_i \Phi(0)$ means $\partial_i \Phi(\vec{r})$ evaluated at $\vec{r} = 0$, and so on.) Equation (6.55) can be written in terms of the external electric field as

$$\Phi(\vec{r}) = \Phi(0) - x_i E_i(0) - \tfrac{1}{2} x_i x_j \partial_i E_j(0) + \cdots . \tag{6.56}$$

Since we are assuming there are no sources for the external electric field within the localised region of interest, it follows that $\partial_i E_i = 0$, and so (6.56) may be re-expressed as

$$\Phi(\vec{r}) = \Phi(0) - x_i E_i(0) - \tfrac{1}{6}(3 x_i x_j - r^2 \delta_{ij}) \partial_i E_j(0) + \cdots . \tag{6.57}$$

(The extra term we have added in is actually zero.)

Using the definitions (6.30) for the multipole moments, we see that when (6.57) is substituted into the expression (6.54) for the "external" energy, it gives

$$\boxed{U_{\rm ext} = Q\, \Phi(0) - p_i\, E_i(0) - \tfrac{1}{6} Q_{ij}\, \partial_i E_j(0) + \cdots . \qquad (6.58)}$$

The first term is the familiar result for the energy of a charge Q in an electrostatic field. The second term is the energy $-\vec{p} \cdot \vec{E}$ of a dipole in an electric field. The third term, which depends on the gradient of the electric field, is the energy of a quadrupole. The higher multipole moments will be associated with higher derivatives of the electric field.

As an application of the result in equation (6.58), we may calculate the interaction energy between a pair of electric dipoles $\vec{p}_1$ and $\vec{p}_2$. Suppose they are located, respectively, at points $\vec{r} = \vec{r}_1$ and $\vec{r} = \vec{r}_2$. From the expression (6.36) for the electric field due to a dipole, we see that the electric field at $\vec{r}_1$ due to a dipole moment $\vec{p}_2$ located at $\vec{r} = \vec{r}_2$ is given by

$$\vec{E}(\vec{r}_1) = \frac{3\vec{n}\,(\vec{n} \cdot \vec{p}_2) - \vec{p}_2}{|\vec{r}_1 - \vec{r}_2|^3} , \tag{6.59}$$

where $\vec{n}$ is the unit vector in the direction from $\vec{r}_2$ to $\vec{r}_1$. From (6.58), we then see that the energy of the dipole $\vec{p}_1$ in this electric field is given by

$$U_{12} = \frac{\vec{p}_1 \cdot \vec{p}_2 - 3(\vec{n} \cdot \vec{p}_1)(\vec{n} \cdot \vec{p}_2)}{|\vec{r}_1 - \vec{r}_2|^3} . \tag{6.60}$$

As one would expect, this expression is completely symmetrical between $\vec{p}_1$ and $\vec{p}_2$.

7 Dielectric Media

7.1 Microscopic description

So far, we have considered situations in which isolated charges or isolated boundary surfaces are present in an otherwise free space (vacuum). In principle, to the extent that classical electromagnetism can be applied at all on the scale of atomic or subatomic particles, one could describe the electrostatic fields in any configuration of matter by means of the Maxwell equations in the form we have been using so far. However, this description would become exteremely unwieldy in cases where one wanted to calculate the electric field inside a lump of matter.

Suppose, for example, we wanted to study the electric field inside an insulator such as salt crystal. At some level one could say that there is a huge array of positive Na ions and negative Cl ions arranged in a lattice. Close to an Na ion the electrostatic potential would grow to huge postive values, whilst near Cl ions the potential would grow to huge negative values. The electric field would, correspondingly, be fluctuating wildly in magnitude and direction, as a function of position within the crystal.

These huge fluctuations would be occurring on the atomic length scale. However, as we know very well, on a *macroscopic* length scale one sees no direct evidence of the wild goings-on at the atomic scale, and it must therefore be possible to *average out* over some appropriate intermediate length scale that lies somewhere between the atomic and the macroscopic scales. Having done this, one should arrive at a microscopic understanding of why the salt crystal has the properties that it does, and in the process one should obtain a phenomenological description of its macroscopic properties.

As far as studying electrostatics in the presence of the salt crystal is concerned, one should arrive at a macroscopic system of modified Maxwell equations that incorporate the phenomenology whose underlying understanding stems from the appropriate averaging-out procedure.

Let us denote the actual, atomic-scale, electric field and charge density by $\vec{\mathcal{E}}$ and ρ'. These will satisfy the standard free-space Maxwell equations

$$\vec{\nabla}\cdot\vec{\mathcal{E}} = 4\pi\rho'\,, \qquad \vec{\nabla}\times\vec{\mathcal{E}} = 0\,. \tag{7.1}$$

From our earlier results, we know that $\vec{\mathcal{E}}$ can be calculated from ρ' using

$$\vec{\mathcal{E}}(\vec{r}) = \int \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}\,\rho'(\vec{r}\,')\,d^3\vec{r}\,'\,. \tag{7.2}$$

This, however, is the field mentioned above that fluctuates wildly, at the atomic length scale, as a function of $\vec{r}$.

We want to consider averaging over scales that are small by macroscopic standards, but which are huge in relation to the atomic length scale. Thus we consider averaged quantities

$$
\begin{aligned}
\langle \vec{\mathcal{E}}(\vec{r}) \rangle &= \frac{1}{\Delta V} \int_{\Delta V} \vec{\mathcal{E}}(\vec{r} + \vec{\xi})\, d^3\vec{\xi}\,, \\
\langle \rho'(\vec{r}) \rangle &= \frac{1}{\Delta V} \int_{\Delta V} \rho'(\vec{r} + \vec{\xi})\, d^3\vec{\xi}\,,
\end{aligned} \tag{7.3}
$$

where ΔV is some macroscopically small volume that nonetheless contains a huge number of atoms.

Note that actually, the ions in the substance are in motion and so one might think that it would be inappropriate to be trying to describe the situation purely by means of electrostatics. However, this motion is random in nature, resulting from thermal excitations, and so once the averaging is performed there will be no significant effects resulting from it. (At least, assuming that the temperatures are not so high that the motion could actually lead to significant electromagnetic radiation.)

Consider a substance composed of atoms or molecules, and first focus attention on the electric field due to the distribution of charge ρ' in the i'th molecule, assumed to be centred at $\vec{r}_i$. The potential outside it will be given by

$$
\phi_i(\vec{r}) = \int \frac{\rho'(\vec{r}\,')}{|\vec{r} - \vec{r}_i - \vec{r}\,'|}\, d^3\vec{r}\,'\,, \tag{7.4}
$$

where the $\vec{r}\,'$ integration is over positions within the molecule. Thus, the electric field will be

$$
\vec{\mathcal{E}}_i(\vec{r}) = -\vec{\nabla} \int \frac{\rho'(\vec{r}\,')}{|\vec{r} - \vec{r}_i - \vec{r}\,'|}\, d^3\vec{r}\,'\,, \tag{7.5}
$$

Applying Taylor's theorem as we did earlier when considering the multipole expansion, we find

$$
\vec{\mathcal{E}}_i(\vec{r}) = -\vec{\nabla} \left[\frac{q_i}{|\vec{r} - \vec{r}_i|} + \vec{p}_i \cdot \vec{\nabla}_i \Big(\frac{1}{|\vec{r} - \vec{r}_i|} \Big) + \cdots \right]\,, \tag{7.6}
$$

where q_i is the total charge of the i'th molecule and $\vec{p}_i$ is its electric dipole moment:

$$
q_i = \int \rho'_i(\vec{r}\,') d^3\vec{r}\,'\,, \qquad \vec{p}_i = \int \vec{r}\,'\, \rho'_i(\vec{r}\,') d^3\vec{r}\,'\,. \tag{7.7}
$$

Note that in (7.6), the symbol $\vec{\nabla}_i$ means the gradient with respect to $\vec{r}_i$, and that the integrations in (7.7) are taken over the volume of the i'th molecule. The higher-order terms in (7.6), involving the quadrupole moment, etc., can be neglected in what follows.

The total microscopic electric field is obtained by summing over the molecules:

$$\vec{\mathcal{E}}(\vec{r}) = -\vec{\nabla} \sum_i \Big[\frac{q_i}{|\vec{r}-\vec{r}_i|} + \vec{p}_i \cdot \vec{\nabla}_i \Big(\frac{1}{|\vec{r}-\vec{r}_i|} \Big) \Big] \,, \tag{7.8}$$

It is helpful at this stage to replace the discrete sum over molecules by a continuous integral, which is achieved in the standard way by representing their charges by a charge density with delta functions, and similarly for their dipole moments:

$$\rho_{\rm mol}(\vec{r}) = \sum_i q_i\, \delta(\vec{r}-\vec{r}_i) \,, \qquad \vec{\pi}_{\rm mol}(\vec{r}) = \sum_i \vec{p}_i\, \delta^3(\vec{r}-\vec{r}_i) \,. \tag{7.9}$$

Thus (7.8) becomes

$$\vec{\mathcal{E}}(\vec{r}) = -\vec{\nabla} \int d^3\vec{r}\,' \Big[\frac{\rho_{\rm mol}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} + \vec{\pi}_{\rm mol}(\vec{r}\,') \cdot \vec{\nabla}' \Big(\frac{1}{|\vec{r}-\vec{r}\,'|} \Big) \Big] \,, \tag{7.10}$$

where the integration of $\vec{r}\,'$ now ranges over the entire volume of the substance.

We now wish to perform the averaging procedure (7.3). Consider first the contribution from the charge term in (7.10); we shall then make an analogous analysis for the dipole term. From (7.3), the charge contribution in (7.10) will give

$$\langle \vec{\mathcal{E}}_{\rm charge}(\vec{r}) \rangle = -\vec{\nabla} \Big[\frac{1}{\Delta V} \int_{\Delta V} d^3\vec{\xi} \int d^3\vec{r}\,' \, \frac{\rho_{\rm mol}(\vec{r}\,')}{|\vec{r}+\vec{\xi}-\vec{r}\,'|} \Big] \,. \tag{7.11}$$

By shifting variables according to

$$\vec{r}\,' \longrightarrow \vec{r}\,' + \vec{\xi}, \tag{7.12}$$

this becomes

$$\langle \vec{\mathcal{E}}_{\rm charge}(\vec{r}) \rangle = -\vec{\nabla} \Big[\frac{1}{\Delta V} \int_{\Delta V} d^3\vec{\xi} \int d^3\vec{r}\,' \, \frac{\rho_{\rm mol}(\vec{r}\,'+\vec{\xi})}{|\vec{r}-\vec{r}\,'|} \Big] \,. \tag{7.13}$$

If we denote by $\langle q_{\rm mol}(\vec{r}\,') \rangle$ the average charge per molecule within the volume ΔV at $\vec{r}\,'$, and if we denote by $N(\vec{r}\,')$ the macroscopic number density of molecules at $\vec{r}\,'$, then we shall have

$$\frac{1}{\Delta V} \int_{\Delta V} d^3\vec{\xi}\, \rho_{\rm mol}(\vec{r}\,'+\vec{\xi}) = N(\vec{r}\,')\, \langle q_{\rm mol}(\vec{r}\,') \rangle \,. \tag{7.14}$$

The contribution (7.13) may then be written as

$$\langle \vec{\mathcal{E}}_{\rm charge}(\vec{r}) \rangle = -\vec{\nabla} \int \frac{N(\vec{r}\,')\, \langle q_{\rm mol}(\vec{r}\,') \rangle}{|\vec{r}-\vec{r}\,'|}\, d^3\vec{r}\,' \,. \tag{7.15}$$

In a similar manner, we introduce the average dipole moment per molecule $\langle \vec{p}_{\rm mol}(\vec{r}\,') \rangle$ at $\vec{r}\,'$, so that

$$\frac{1}{\Delta V} \int_{\Delta V} d^3\vec{\xi}\, \vec{\pi}_{\rm mol}(\vec{r}\,'+\vec{\xi}) = N(\vec{r}\,')\, \langle \vec{p}_{\rm mol}(\vec{r}\,') \rangle \,. \tag{7.16}$$

The total average of the electric field (7.10) is therefore given by

$$\langle\vec{\mathcal{E}}(\vec r)\rangle = -\vec\nabla\int N(\vec r\,')\Big[\frac{\langle q_{\rm mol}(\vec r\,')\rangle}{|\vec r-\vec r\,'|} + \langle\vec p_{\rm mol}(\vec r\,')\rangle\cdot\vec\nabla'\Big(\frac{1}{|\vec r-\vec r\,'|}\Big)\Big]\,d^3\vec r\,'\,. \tag{7.17}$$

Taking the divergence of (7.17), and recalling that $\nabla^2|\vec r-\vec r\,'|^{-1} = -4\pi\delta^3(\vec r-\vec r\,')$, gives

$$\vec\nabla\cdot\langle\vec{\mathcal{E}}(\vec r)\rangle = 4\pi\int N(\vec r\,')\Big[\langle q_{\rm mol}(\vec r\,')\rangle\delta^3(\vec r-\vec r\,') + \langle\vec p_{\rm mol}(\vec r\,')\rangle\cdot\vec\nabla'\,\delta^3(\vec r-\vec r\,')\Big]\,d^3\vec r\,'\,. \tag{7.18}$$

Performing the integrations[17] then gives

$$\vec\nabla\cdot\langle\vec{\mathcal{E}}(\vec r)\rangle = 4\pi N(\vec r)\langle q_{\rm mol}(\vec r)\rangle - 4\pi\vec\nabla\cdot\Big(N(\vec r)\langle\vec p_{\rm mol}(\vec r)\rangle\Big)\,. \tag{7.19}$$

Equation (7.19) is the phenomenological replacement for the Maxwell equation $\vec\nabla\cdot\vec E = 4\pi\rho$, in the case where we average out over molecules in a dielectric medium. Effectively, the usual charge density on the right-hand side is replaced by the sum of two terms. The first is the average charge per unit volume of the molecules themselves, and the second is the average polarisation charge per unit volume. The equation can be rewritten by taking this second term over to the left-hand side, giving

$$\vec\nabla\cdot\Big(\langle\vec{\mathcal{E}}(\vec r)\rangle + 4\pi N(\vec r)\langle\vec p_{\rm mol}(\vec r)\rangle\Big) = 4\pi N(\vec r)\langle q_{\rm mol}(\vec r)\rangle\,. \tag{7.20}$$

We may now define macroscopic quantities as follows:

$$\vec E\equiv\langle\vec{\mathcal{E}}\rangle\,,\qquad \vec P\equiv N\,\langle\vec p_{\rm mol}\rangle\,,\qquad \rho\equiv N\langle q_{\rm mol}\rangle\,,\qquad \vec D\equiv\vec E+4\pi\vec P\,. \tag{7.21}$$

$\vec E$ is called the electric field; $\vec P$ is the *polarisation* (i.e. the electric dipole moment per unit volume); $\vec D$ is called the *displacement*; and ρ is the charge density.

If there are various different types of molecules or atoms comprising the medium, and if, furthermore, there are additional "external" charges present, then $\vec P$ and ρ defined above admit a natural generalisation, obtained by summing over all the contributions:

$$\vec P = \sum_i N_i\langle\vec p_i\rangle\,,\qquad \rho = \rho_{\rm ext}+\sum_i N_i\langle q_i\rangle\,. \tag{7.22}$$

Commonly, the molecules themselves are electrically neutral, and so the charge density ρ is just given by the external charge density $\rho_{\rm ext}$.

In terms of these definition, we see that the effective phenomenological Maxwell equations describing electrostatics in the presence of dielectric media will be

$$\vec\nabla\cdot\vec D = 4\pi\rho\,,\qquad \vec\nabla\times\vec E = 0\,. \tag{7.23}$$

[17]Recall that $\int f(y)\delta'(y-x)dy = -\int f'(y)\delta(y-x)dy = -f'(x)$, where a prime denotes the derivative with respect to the argument, and so analogously $\int\vec f(\vec r\,')\cdot\vec\nabla'\delta^3(\vec r-\vec r\,')d^3\vec r\,' = -\vec\nabla\cdot\vec f(\vec r)$ for any vector field $\vec f(\vec r)$.

(The second equation follows by taking the curl of (7.17).)

It is evident from the second equation in (7.23) that $\vec{E}$ can still be written in terms of a potential,

$$\vec{E} = -\vec{\nabla}\phi\,. \tag{7.24}$$

Conversely, by integrating $\vec{E}$ along a path, one still obtains the potential difference between the initial and final points:

$$\int_A^B \vec{E}\cdot d\vec{\ell} = -\int \vec{\nabla}\phi\cdot d\vec{\ell} = \phi(A) - \phi(B)\,. \tag{7.25}$$

The potential ϕ can be solved for, in terms of the macroscopic quantities, as

$$\phi(\vec{r}) = \int d^3\vec{r}\,' \,\Big[\frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} + \vec{P}(\vec{r}\,')\cdot\vec{\nabla}'\Big(\frac{1}{|\vec{r}-\vec{r}\,'|}\Big)\Big]\,. \tag{7.26}$$

(See equation (7.17).)

7.2 Examples of dielectric media

The polarisation $\vec{P}$ of the molecules or atoms of which a medium is composed is related to the local electric field in their neighbourhood. Typically, if no external electric field is applied the polarisation will be zero. To a very good approximation, the polarisation will increase linearly with increasing applied electric field. Thus, using the index notation, and Einstein summation convention, for vector and tensor fields, we can say that

$$P_i = a_{ij}\,E_j\,, \tag{7.27}$$

where the tensor a_{ij} is independent of $\vec{E}$. It is known as the *electric susceptibility tensor*. In general, in a medium that is not homogeneous, it can depend on position. If the medium is homogeneous (i.e. it is the same throughout), then a_{ij} will be a constant tensor.

If the medium is in addition *isotropic*, then the polarisation will be parallel to the electric field.[18] In such cases, we shall have

$$a_{ij} = \chi_e\,\delta_{ij}\,, \tag{7.28}$$

where χ_e is called the *electric susceptibility* of the medium. Then we shall have

$$\vec{P} = \chi_e\,\vec{E}\,. \tag{7.29}$$

[18]There are substances, such as calcite or quartz, which ar non-isotropic, and for these the electric susceptibility is necessarily a tensor.

From the definition of $\vec{D}$ in (7.21), we then have

$$\vec{D} = \epsilon\,\vec{E}\,, \tag{7.30}$$

where

$$\epsilon \equiv 1 + 4\pi\,\chi_e \tag{7.31}$$

is called the *dielectric constant* of the medium.

If we assume, therefore, a homogeneous isotropic medium, we shall have (7.30) for some constant ϵ, and so the effective Maxwell equation for $\vec{D}$ given in (7.23) implies

$$\vec{\nabla}\cdot\vec{E} = \frac{4\pi}{\epsilon}\,\rho\,. \tag{7.32}$$

This means that, compared with the free-space solutions that we discussed in previous chapters, the solutions in the presence of the medium will be exactly the same, except that $\vec{E}$ is scaled down by a factor of $1/\epsilon$. (The dielectric constant is usually greater than 1.)

For example, suppose we place a point charge q in an infinite medium of dielectric constant ϵ. Choosing coordinates so that the charge is located at the origin, it follows that the electrostatic potential will be given by

$$\phi = \frac{q}{\epsilon\,r}\,, \tag{7.33}$$

and therefore that the electric field $\vec{E} = -\vec{\nabla}\phi$ is given by

$$\vec{E} = \frac{q\,\vec{r}}{\epsilon\,r^3}\,. \tag{7.34}$$

The physical interpretation of what is happening in the medium is that when an external electric field is applied, the charges in the atoms or molecules tend to be displaced slightly, in such a way that the positive charges are pulled in the direction of the lower electrostatic potential, whilst the negative charges are pulled in the direction of the larger potential. This induces a dipole moment that points in the opposite direction to the electric field. In other words, the the dipole moment that develops when external charges are introduced is such as to tend to *oppose* the external charges.

7.3 Boundary-value problems with dielectric interfaces

In section 1.4, we derived the boundary conditions that must hold at an interface between two media. In the case of electrostatics, with dielectric media, the relevant conditions are (see eqn 1.32))

$$\vec{n}\cdot(\vec{D}_2 - \vec{D}_1) = 4\pi\sigma\,, \qquad \vec{n}\times(\vec{E}_2 - \vec{E}_1) = 0\,, \tag{7.35}$$

where the subscripts 1 and 2 indicate the fields on either side of the interface, in medium 1 and medium 2 respectively; $\vec{n}$ is the unit normal vector at the interface, pointing from medium 1 to medium 2; and σ is the surface charge density at the interface. Note that σ does *not* include polarisation charges.

The two conditions in (7.35) came from integrating the phenomenological Maxwell equation $\vec{\nabla}\cdot\vec{D} = 4\pi\rho$ in (7.23) over a Gaussian pillbox straddling the interface, and integrating $\vec{\nabla}\times\vec{E} = 0$ around a loop straddling the interface, respectively. The first condition says that there is a discontinuity in the normal component of $\vec{D}$, given by 4π times the surface charge density. The second condition says that the tangential components of the electric field $\vec{E}$ must be continuous across the interface.

We now consider several example electrostatic problems involving dielectric media.

7.3.1 Method of images for two semi-infinite dielectrics

Suppose that a medium with dielectric constant ϵ_1 fills the entire half-space $z > 0$, and that a medium with dielectric constant ϵ_2 fills the other half of space, for $z < 0$. Thus we have an interface in the plane $z = 0$. Suppose that a point charge q is placed at the point $\vec{r} = (0, 0, d)$, in dielectric 1. The problem is to solve for the electric field everywhere.

From (7.23), the equations to be solved are

$$
\begin{aligned}
\epsilon_1\,\vec{\nabla}\cdot\vec{E} &= 4\pi\rho = 4\pi q\delta(x)\delta(y)\delta(z-d)\,, \qquad z > 0\,,\\
\epsilon_2\,\vec{\nabla}\cdot\vec{E} &= 0\,, \qquad z < 0\,,\\
\vec{\nabla}\times\vec{E} &= 0\,, \qquad \text{all } z\,.
\end{aligned}
\tag{7.36}
$$

Since $\vec{\nabla}\times\vec{E} = 0$, we can write the electric field in terms of a scalar potential, $\vec{E} = -\vec{\nabla}\phi$, as always. The boundary conditions (7.35) imply that

$$
\begin{aligned}
\epsilon_1 E_z(x,y,0^+) &= \epsilon_2 E_z(x,y,0^-)\,,\\
E_x(x,y,0^+) &= E_x(x,y,0^-)\,,\\
E_y(x,y,0^+) &= E_y(x,y,0^-)\,,
\end{aligned}
\tag{7.37}
$$

where 0^+ indicates 0 approached from above (and so $z = 0^+$ means that z is in medium 1), and 0^- indicates 0 approached from below (medium 2).

In section (2.1), we solved the problem of a point charge q above an infinite planar conductor by introducing an image charge $-q$ at the mirror-image reflection point below the conductor. Here, we can apply a similar image-charge technique. However, because of the fact that we now have dielectric media rather than a conducting plane, the necessary procedure is less intuitively clear. A crucial point, though, is that, as we saw earlier, there is a *uniqueness theorem* in electrostatics which means that if, by hook or by crook, we obtain a solution to the equations that *works*, then we are guaranteed that it is the unique and correct solution to the problem. (By "works," we mean that is satisfies the equations and the given boundary conditions.)

The approach we shall take here in solving the problem will be to try some plausible guesses, and discover that they work. This then means that the problem is solved.

To proceed, we must choose suitable "trial solutions" in each of the two regions $z > 0$ and $z < 0$.

Considering the region $z > 0$ first, is natural, by analogy with the usual method of images procedure, to guess that here we should introduce an image charge at the mirror-reflection of the location of the actual charge q; i.e. introduce an image charge at $\vec{r} = (0, 0, -d)$. However, clearly the *value* of the image charge will no longer in general be simply $-q$.[19] Thus for $z > 0$ we shall look for a solution of the form

$$\phi = \frac{q}{\epsilon\, R_1} + \frac{\alpha}{R_2}\,, \tag{7.38}$$

where α is an unknown constant, and

$$R_1^2 = x^2 + y^2 + (z-d)^2\,, \qquad R_2^2 = x^2 + y^2 + (z+d)^2\,. \tag{7.39}$$

It is convenient to reparameterise α by writing $\alpha = q'/\epsilon_1$, so the trial solution for $z > 0$ will be of the form

$$\boxed{\phi = \frac{1}{\epsilon_1}\Big(\frac{q}{R_1} + \frac{q'}{R_2}\Big)\,, \qquad z > 0\,,} \tag{7.40}$$

where q' is an as-yet unknown constant.

The expression (7.40) is intended to apply *only* to observation points in the upper half-space. We also need an expression for the potential that is valid instead in the lower half-space, $z < 0$. Here, we make the plausible guess that it takes the form

$$\phi = \frac{\beta}{R_1}\,, \tag{7.41}$$

where β is an unknown constant. In other words, we are guessing that the potential in the region $z < 0$ has the form that would result from a charge placed at the location of

[19]This is obvious from the fact that if $\epsilon_1 = \epsilon_2 = 1$, so that the original charge q is in an infinite free space, no image charge at all is needed. On the other hand, if $\epsilon_1 = 1$ and $\epsilon_2 = \infty$, we are back to the situation of the infinite planar conductor, for which we know the image charge should be $-q$.

the original physical charge q, but with some as-yet unknown coefficient. (Note that the trial expression for ϕ in the region $z < 0$) should certainly not contain any charges that are themselves located in the region $z < 0$; if there were any such charge, it would imply (incorrectly) that the potential in $z < 0$ would diverge somewhere.) It is convenient to reparameterise β as $\beta = q''/\epsilon_2$, and so in the region $z < 0$ we choose the trial solution

$$\boxed{\phi = \frac{q''}{\epsilon_2 R_1}\,, \qquad z < 0\,,} \tag{7.42}$$

where q'' is an as-yet unknown constant.

In case the reader is feeling doubtful at this point, recall again that the *only* thing that matters in the end is that the proposed solution should (a) satisfy the equations (7.23), and (b) satisfy the boundary conditions (7.37) and at infinity. Thus, we take (7.40) and (7.42) as the guess[20], and now check to see if we can choose q' and q'' so that the boundary conditions (7.37) are satisfied.

To impose the boundary conditions, we need to calculate the derivatives of $1/R_1$ and $1/R_2$ with respect to x, y and z, evaluated at the interface $z = 0$. We see that

$$\begin{aligned}
\frac{\partial}{\partial x}\Big(\frac{1}{R_1}\Big)\Big|_{z=0} = \frac{\partial}{\partial x}\Big(\frac{1}{R_2}\Big)\Big|_{z=0} &= -\frac{x}{(x^2+y^2+d^2)^{3/2}}\,,\\
\frac{\partial}{\partial y}\Big(\frac{1}{R_1}\Big)\Big|_{z=0} = \frac{\partial}{\partial y}\Big(\frac{1}{R_2}\Big)\Big|_{z=0} &= -\frac{y}{(x^2+y^2+d^2)^{3/2}}\,,\\
\frac{\partial}{\partial z}\Big(\frac{1}{R_1}\Big)\Big|_{z=0} = -\frac{\partial}{\partial z}\Big(\frac{1}{R_2}\Big)\Big|_{z=0} &= \frac{d}{(x^2+y^2+d^2)^{3/2}}\,.
\end{aligned} \tag{7.43}$$

Thus we see that the boundary conditions (7.37) imply

$$\begin{aligned}
q'' &= q - q'\,,\\
\frac{1}{\epsilon_1}(q+q') &= \frac{1}{\epsilon_2}\,q''\,.
\end{aligned} \tag{7.44}$$

The solution to these equations is

$$\boxed{q' = \frac{\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2}\,q\,, \qquad q'' = \frac{2\epsilon_2}{\epsilon_1+\epsilon_2}\,q\,.} \tag{7.45}$$

Observe that (7.45) clearly makes sense in two limiting cases. Firstly, consider the case when $\epsilon_1 = \epsilon_2$, meaning that all of space is filled with a single dielectric medium. Obviously, there is no interface at all in this case, and so no image charge is needed. And indeed, (7.45) gives $q' = 0$. Furthermore, the formula for the potential that is valid when $z < 0$ should be the same as the one valid for $z > 0$. And indeed, $q'' = q$ in this case.

[20]Or *ansatz*, to use a slightly more scientific-sounding word that means a guess.

Secondly, consider the limit when $\epsilon_2 = \infty$. This means that effectively, the region $z < 0$ is just a semi-infinite slab of conductor. And indeed, we see that the expression for the potential in the region $z > 0$ just reduces to the original image-charge result for an infinite conductor, as discussed in section (2.1).

In the derivation above, we have been careful not to attach any direct physical significance to the "fictitious" charges q' and q'' that were introduced in order to obtain the correct solution for the electrostatic potential in the two half-spaces $z > 0$ and $z < 0$. Observe, indeed, that it was never necessary to give them any direct interpretation; they are just coefficients of terms we introduced in the candidate expressions for the potential in the two half-spaces, and by "lucky chance," it turned out that by tuning q' and q'' appropriately, we were able to solve the problem.

By definition, the *polarisation surface charge density* is given by

$$\sigma_{\rm pol} = -\vec{\nabla} \cdot \vec{P} \,. \tag{7.46}$$

Inside each dielectric, we have (see (7.29))

$$\vec{P} = \chi_e \vec{E} \,, \tag{7.47}$$

where $\chi_e = (\epsilon_1 - 1)/(4\pi)$ in medium 1, and $\chi_e = (\epsilon_2 - 1)/(4\pi)$ in medium 2. Since $\vec{\nabla} \cdot \vec{E} = 0$ everywhere in each region (except for the location of the physical charge q), it follows that the polarisation charge density is zero inside each medium.

However, there is a discontinuity in χ_e at the interface $z = 0$; it jumps by an amount

$$\Delta\chi_e = \frac{\epsilon_1 - \epsilon_2}{4\pi} \,. \tag{7.48}$$

This implies that there is a polarisation charge density $\sigma_{\rm pol}$ on the boundary surface, given by

$$\sigma_{\rm pol} = -(\vec{P}_2 - \vec{P}_1) \cdot \vec{n} \,, \tag{7.49}$$

where $\vec{n}$ is the unit normal vector pointing from medium 1 to medium 2. $\vec{P}_1$ and $\vec{P}_2$ are the polarisations in the two media, at the surface $z = 0$.

With

$$\vec{P}_i = \frac{\epsilon_i - 1}{4\pi} \vec{E} = -\frac{\epsilon_i - 1}{4\pi} \nabla\phi \,, \tag{7.50}$$

for $i = 1$ or 2 in the two media, we see that

$$\sigma_{\rm pol} = -\frac{q\,(\epsilon_2 - \epsilon_1)}{2\pi\epsilon_1(\epsilon_2 + \epsilon_1)} \, \frac{d}{(x^2 + y^2 + d^2)^{3/2}} = \frac{q'}{2\pi\epsilon_1} \, \frac{d}{(x^2 + y^2 + d^2)^{3/2}} \,. \tag{7.51}$$

If this is integrated over the entire $z = 0$ plane, it gives a total polarisation charge

$$Q_{\rm pol} = q' \,. \tag{7.52}$$

7.3.2 Dielectric sphere

Consider a sphere of radius a and dielectric constant ϵ, placed in an originally-uniform electric field $\vec{E}$. Thus we imagine that prior to introducing the sphere, there is a uniform electric field $\vec{E}$ directed along the z direction. Let its magnitude be denoted by E_0. Thus in the absence of the dielectric sphere, we have

$$\phi = -E_0\, z\,, \qquad \Rightarrow \qquad \vec{E} = (0, 0, E_0)\,. \tag{7.53}$$

Note that in spherical polar coordinates we therefore have

$$\phi = -E_0\, r\cos\theta\,. \tag{7.54}$$

After we have introduced the spherical dielectric, we shall take (7.54) as the asymptotic form of the potential at large r. (This implements the notion of the "originally uniform electric field.")

Now we introduce the dielectric sphere. For convenience, we choose our coordinates so that it is centred on the origin of spherical polar coordinates. There will be no free charges either inside or outside the sphere, and so in each region we must have $\nabla^2\phi = 0$. In the regions inside and outside the sphere we therefore must have

$$\begin{aligned} r < a: &\qquad \phi_<(r,\theta) = \sum_{\ell\ge 0} A_\ell\, r^\ell\, P_\ell(\cos\theta)\,,\\ r > a: &\qquad \phi_>(r,\theta) = \sum_{\ell\ge 0} \Big(\widetilde{A}_\ell\, r^\ell + B_\ell\, r^{-\ell-1}\Big)\, P_\ell(\cos\theta)\,. \end{aligned} \tag{7.55}$$

Note that normally, we would have assumed in the large-r region that only the terms with inverse powers of r should be included, since the potential is normally assumed to fall off as r goes to infinity. However, here we are going to impose the boundary condition (7.54) at large r. This means, since $P_1(\cos\theta) = \cos\theta$, that we must take

$$\widetilde{A}_1 = -E_0, \qquad\qquad \widetilde{A}_\ell = 0 \quad \text{for} \quad \ell \ne 1\,. \tag{7.56}$$

We must now match the two expressions for the potential at the interface at $r = a$, using the boundary conditions (7.35). These give

$$\begin{aligned} -\frac{1}{a}\,\frac{\partial\phi_<(r,\theta)}{\partial\theta}\Big|_{r=a} &= -\frac{1}{a}\,\frac{\partial\phi_>(r,\theta)}{\partial\theta}\Big|_{r=a}\,,\\ -\epsilon\,\frac{\partial\phi_<(r,\theta)}{\partial r}\Big|_{r=a} &= -\frac{\partial\phi_>(r,\theta)}{\partial r}\Big|_{r=a}\,. \end{aligned} \tag{7.57}$$

(The first equation comes from matching the tangential components of $\vec{E}$ across the boundary, and the second from matching the normal component of $\vec{D}$.)

The first equation in (7.57) implies

$$A_1 = -E_0 + B_1\, a^{-3}\,, \qquad\qquad A_\ell = B_\ell\, a^{-2\ell-1} \quad \text{for} \quad \ell \neq 1\,. \tag{7.58}$$

The second equation in (7.57) implies

$$\epsilon A_1 = -E_0 - 2B_1\, a^{-3}\,, \qquad\qquad \ell\epsilon A_\ell = -(\ell+1) B_\ell\, a^{-2\ell-1} \quad \text{for} \quad \ell \neq 1\,. \tag{7.59}$$

The $\ell \neq 1$ equations in (7.58) and (7.59) imply that $A_\ell = B_\ell = 0$ for all $\ell \neq 1$. The $\ell = 1$ equations imply

$$A_1 = -\Big(\frac{3}{\epsilon+2}\Big)\, E_0\,, \qquad B_1 = \Big(\frac{\epsilon-1}{\epsilon+2}\Big)\, a^3\, E_0\,. \tag{7.60}$$

Thus the potentials inside and outside the sphere are given by

$$\boxed{\begin{aligned} \phi_<(r,\theta) &= -\Big(\frac{3}{\epsilon+2}\Big)\, E_0\, r\, \cos\theta\,, \\ \phi_>(r,\theta) &= -E_0\, r\, \cos\theta + \Big(\frac{\epsilon-1}{\epsilon+2}\Big)\, E_0\, \frac{a^3}{r^2}\, \cos\theta\,. \end{aligned}} \tag{7.61}$$

The potential inside the sphere implies that the electric field is uniform for $r < a$, and parallel to the external applied field. Its magnitude is given by

$$E_< = \frac{3}{\epsilon+2}\, E_0\,. \tag{7.62}$$

The potential outside the sphere is a sum of two terms. The first just gives the original uniform electric field E_0 (see (7.54)). The second term in the expression for $\phi_>$ in (7.61) can be written as

$$\Big(\frac{\epsilon-1}{\epsilon+2}\Big)\, a^3\, E_0\, \frac{\vec z\cdot\vec r}{r^3}\,, \tag{7.63}$$

where $\vec z = (0,0,1)$ is the unit vector along the z axis. This term can be recognised (see (6.31) as the potential due to a dipole of moment

$$\vec p = \Big(\frac{\epsilon-1}{\epsilon+2}\Big)\, a^3\, \vec E_0\,, \tag{7.64}$$

pointing along the direction of the applied electric field (i.e. along the z axis).

Inside the sphere, the polarisation $\vec P$ is given by

$$\vec P = \Big(\frac{\epsilon-1}{4\pi}\Big)\, \vec E = \frac{3}{4\pi}\, \Big(\frac{\epsilon-1}{\epsilon+2}\Big)\, \vec E_0\,. \tag{7.65}$$

Since this is constant inside the sphere, we see that its volume integral is given by

$$\int_{\text{sphere}} \vec P = \tfrac{4}{3}\, \pi a^3\, \vec P = \Big(\frac{\epsilon-1}{\epsilon+2}\Big)\, a^3\, \vec E_0\,. \tag{7.66}$$

Comparing with (7.64), we see that the dipole moment $\vec{p}$ is nothing but the volume integral of the polarisation $\vec{P}$ inside the sphere.

From (7.49), it can be seen that the polarisation surface-charge density on the surface of the sphere is given by $\vec{P} \cdot \vec{r}/r$, and so

$$\sigma_{\rm pol} = \frac{3}{4\pi} \left(\frac{\epsilon - 1}{\epsilon + 2}\right) E_0 \cos\theta \,. \tag{7.67}$$

This can be understood as generating an internal electric field that tends to oppose the applied field, thus reducing the electric field inside the sphere to the value given by (7.62).

7.4 Electrostatic energy in dielectric media

In free space, we saw that the electrostatic self energy of a charge distribution ρ was given by

$$U = \tfrac{1}{2} \int \rho(\vec{r})\phi(\vec{r}) d^3\vec{r}\,. \tag{7.68}$$

Using $\vec{\nabla} \cdot \vec{E} = 4\pi\rho$ and $\vec{E} = -\vec{\nabla}\phi$, this could be rewritten as

$$U = \frac{1}{8\pi} \int |\vec{E}|^2\, d^3\vec{r}\,. \tag{7.69}$$

We may derive the analogous expression in a dielectric medium as follows. Suppose that a charge distribution ρ gives rise to the potential ϕ, and that we then make an infinitesimal change $\delta\rho$ to the charge distribution. The work done in making this change will be given by

$$\delta U = \int \delta\rho\, \phi d^3\vec{r}\,. \tag{7.70}$$

Using $\vec{\nabla} \cdot \vec{D} = 4\pi\rho$, we have $\delta\rho = (\vec{\nabla} \cdot \delta\vec{D})/(4\pi)$, and so

$$\delta U = \frac{1}{4\pi} \int \phi(\vec{\nabla} \cdot \delta\vec{D})\, d^3\vec{r} = -\frac{1}{4\pi} \int (\vec{\nabla}\phi) \cdot \delta\vec{D}\, d^3\vec{r}\,, \tag{7.71}$$

where we have dropped the boundary term in the integration by parts, on account of the fact that the charge distribution is taken to be localised in some finite region. Using $\vec{E} = -\vec{\nabla}\phi$, we therefore have

$$\delta U = \frac{1}{4\pi} \int \vec{E} \cdot \delta\vec{D}\, d^3\vec{r}\,. \tag{7.72}$$

The energy of the charge distribution can thus be calculated by integrating up from $\vec{D} = 0$ to its final value. In general, this integration could be non-trivial, since $\vec{D}$ and $\vec{E}$ could be related in a complicated way.

Suppose, however, that the medium is *linear*, in the sense that $\vec{D}$ is *proportional* to $\vec{E}$. This is true, for example, if $\vec{D} = \epsilon\vec{E}$. For a linear medium, we must have that

$$\vec{E} \cdot \delta\vec{D} = (\delta\vec{E}) \cdot \vec{D}\,, \tag{7.73}$$

and hence we can write

$$\vec{E}\cdot\delta\vec{D} = \tfrac{1}{2}\delta(\vec{E}\cdot\vec{D})\,. \tag{7.74}$$

It is then straightforward to integrate up from $\vec{D}=0$ to its final value, and so we find from (7.72) that

$$\boxed{U = \frac{1}{8\pi}\int \vec{E}\cdot\vec{D}\, d^3\vec{r}. \qquad (7.75)}$$

Interestingly, if we now use $\vec{E} = -\vec{\nabla}\phi$ in (7.75), integrate by parts, and then use $\vec{\nabla}\cdot\vec{D} = 4\pi\rho$, we obtain

$$U = \tfrac{1}{2}\int \rho(\vec{r})\phi(\vec{r}) d^3\vec{r}, \tag{7.76}$$

which is the same as the free-space result. However, it should be emphasised that in deriving (7.75) it was necessary to use the assumption of *linearity* of the relation between $\vec{D}$ and $\vec{E}$, and so although (7.76) holds in dielectric media with a linear response, it does not hold if the relation between $\vec{D}$ and $\vec{E}$ is non-linear.

Suppose that a distribution of charge $\rho_0(x)$ inside a medium of dielectric constant ϵ_0 gives rise to an electric field $\vec{E}_0$. We may allow ϵ_0 to be position dependent in the medium. We know from the discussion above that this configuration will have electrostatic energy U given by

$$U_0 = \frac{1}{8\pi}\int \vec{E}_0\cdot\vec{D}_0\, d^3\vec{r}, \tag{7.77}$$

with $\vec{D}_0 = \epsilon_0\vec{E}_0$.

While keeping the charge distribution fixed, suppose now that a piece of dielectric material of volume V and dielectric constant ϵ_1 is introduced into the field. This has the effect of making the dielectric constant ϵ equal to ϵ_1 inside V, while it remains at its original value ϵ_0 outside V. We can think of $\epsilon = \epsilon(\vec{r})$, with $\epsilon(\vec{r})$ changing smoothly but rapidly from ϵ_1 inside V to ϵ_0 outside V. The electric field is now $\vec{E}$. The energy is given by

$$U_1 = \frac{1}{8\pi}\int \vec{E}\cdot\vec{D}\, d^3\vec{r}, \tag{7.78}$$

where $\vec{D} = \epsilon(\vec{r})\,\vec{E}$.

The change in energy, $\Delta U = U_1 - U_0$, is given by

$$\Delta U = \frac{1}{8\pi}\int (\vec{E}\cdot\vec{D} - \vec{E}_0\cdot\vec{D}_0) d^3\vec{r}. \tag{7.79}$$

Clearly this can be written as

$$\Delta U = \frac{1}{8\pi}\int (\vec{E}\cdot\vec{D}_0 - \vec{E}_0\cdot\vec{D}) d^3\vec{r} + \frac{1}{8\pi}\int (\vec{E}+\vec{E}_0)\cdot(\vec{D}-\vec{D}_0) d^3\vec{r}. \tag{7.80}$$

Since $\vec{\nabla}\times\vec{E}=0$ and $\vec{\nabla}\times\vec{E}_0=0$, it follows that we can write $\vec{E}+\vec{E}_0=\vec{\nabla}\psi$ for some scalar function ψ, and so the second integral gives

$$\frac{1}{8\pi}\int\vec{\nabla}\psi\cdot(\vec{D}-\vec{D}_0)d^3\vec{r}=-\frac{1}{8\pi}\int\psi\vec{\nabla}\cdot(\vec{D}-\vec{D}_0)d^3\vec{r}\,. \qquad (7.81)$$

Since we assumed that the charge distribution ρ_0 was unaltered by the introduction of the dielectric medium ϵ_1, it follows that $\vec{\nabla}\cdot\vec{D}=\vec{\nabla}\cdot\vec{D}_0=4\pi\rho_0$, and hence the integral (7.81) gives zero. Thus we have

$$\Delta U=\frac{1}{8\pi}\int(\vec{E}\cdot\vec{D}_0-\vec{E}_0\cdot\vec{D})d^3\vec{r}\,. \qquad (7.82)$$

Since $\vec{D}=\epsilon_0\vec{E}$ at points outside V, it follows that the integration in (7.82) need be performed only within V, and so we get

$$\Delta U=-\frac{1}{8\pi}\int_V(\epsilon_1-\epsilon_0)\vec{E}\cdot\vec{E}_0 d^3\vec{r}\,. \qquad (7.83)$$

If the original medium was in fact just free space, so that $\epsilon_0=1$, then it follows from $\vec{D}=\vec{E}+4\pi\vec{P}$, and $\vec{D}=\epsilon\vec{E}$, that (7.83) becomes

$$\Delta U=-\tfrac{1}{2}\int_V\vec{P}\cdot\vec{E}_0\,d^3\vec{r}\,, \qquad (7.84)$$

where $\vec{P}$ is the polarisation of the dielectric material. From this we see that if a dielectric object has polarisation $\vec{P}$ when placed in an electric field $\vec{E}_0$ that is generated by fixed sources, then it will have an energy density

$$w=-\tfrac{1}{2}\vec{P}\cdot\vec{E}_0\,. \qquad (7.85)$$

This expression is analogous to the contribution in (6.58) giving the energy of a dipole $\vec{p}$ in an electric field. The reason for the factor of $\frac{1}{2}$ in (7.85) is that, rather than being like a *fixed* dipole moment, the polarisation $\vec{P}$ in (7.85) is itself *caused* by the presence of the electric field $\vec{E}_0$.

The implication of the above calculation is that a dielectric object (with $\epsilon>1$) will tend to be drawn into a region of increasing electric field $\vec{E}_0$. The force can be calculated by making a virtual infinitesimal displacement $\delta\vec{r}$ of the object, leading to a change δU in the energy. Since the charges are held fixed, there is no change in the external energy, and so the change in the field energy can be interpreted as a change in the potential energy of the object. There will therefore be a force acting on the object, given by

$$\vec{F}=-\vec{\nabla}U\,. \qquad (7.86)$$

It is important to emphasise that this is *when we hold the charges that generate the electric field fixed.*

An alternative situation, and in fact one that in practice is more likely to arise, is that the electric field is generated by means of electrodes or plates that are held at a *fixed potential*, rather than with *fixed charge.* This would arise if the electrodes are attached to a battery or some other constant-voltage source. The situation is then like the classic problem in elementary electrostatics, in which one calculates the atttractive force between the plates of a parallel plate capacitor. The easier calculation is when one assumes that the charge on the plates is fixed (achieved by connecting the plates to the battery and then disconnecting the battery). The force is calculated by working out the energy change when a small virtual displacement is made. A more complicated way of reaching the same conclusion is to keep the battery connected during the virtual displacement, so that now the potential, instead of the charge, is held fixed. Naively, one now arrives at the (incorrect) conclusion that the potential energy is *increased* if the plates are moved together, implying a repulsive force that is equal in magnitude to the previously-calculated attractive force. Of course, what the constant-potential argument has neglected is that the battery has to do work *in order to keep the potential fixed* during the virtual displacement. In fact it does twice as much work as the field energy-change, and so when the changes in field energy *plus* work done by the battery are added, the net energy change is the same in magnitdude *and in sign* as the net energy change in the constant-charge virtual displacement. Thus the force between the capacitor plates comes out to be the same in magnitude and sign, regardless of whether the potential difference or the charge is held fixed.

Of course it is clear from common sense that this must be the case. A fixed capacitor (assumed to be ideal, with no leakage) holds the same potential difference whether the battery is connecetd or not, and so the force cannot possibly suddenly switch in sign depending on whether the battery is connected or not.

In the same way, one can do the analogous constant-potential calculation for a dielectric object being drawn into an electric field. The final conclusion is that it is drawn in, with the same force, whether the battery is connected or not.

8 Magnetostatics

We now turn to the magnetic analogue of electrostatics, namely magnetostatics. This describes the situation when everything is independent of time, and only magnetic fields and electric currents are present.

8.1 Ampère's law and the Biot-Savat law

From the general form of Maxwell's equations (1.3), we see that the only non-trivial equations to be considered are

$$\boxed{\vec\nabla\cdot\vec B=0\,,\qquad \vec\nabla\times\vec B=\frac{4\pi}{c}\,\vec J\,.} \tag{8.1}$$

(Recall that we are using Gaussian units, and that c is the speed of light.)

In view of $\vec\nabla\cdot\vec B=0$, we can write $\vec B$ in terms of a vector potential $\vec A$, as

$$\vec B=\vec\nabla\times\vec A\,. \tag{8.2}$$

The choice of vector potential is not unique, and we can perform a *gauge transformation* of the form

$$\vec A\longrightarrow\vec A+\vec\nabla\lambda\,, \tag{8.3}$$

where λ is any function of $\vec r$, since obviously when (8.3) is substituted into (8.2), the term involving λ gives zero.

We may employ the *gauge invariance* of the system under the transformations (8.3) in order to impose a convenient *gauge condition* called the *Coulomb gauge*

$$\vec\nabla\cdot\vec A=0\,. \tag{8.4}$$

This can be seen by supposing that we started with a gauge potential $\widetilde{\vec A}$ that was not in the Coulomb gauge, and transforming it according to (8.3) to give

$$\vec A=\widetilde{\vec A}+\vec\nabla\lambda\,, \tag{8.5}$$

where $\vec A$ *is* in Coulomb gauge; i.e. $\vec\nabla\cdot\vec A=0$. Taking the divergence of (8.5), we therefore see that

$$\nabla^2\lambda=-\vec\nabla\cdot\widetilde{\vec A}\,. \tag{8.6}$$

This Poisson equation can always be solved; in fact, we are familiar with its solution from solving the same equation in electrostatics (i.e. $\nabla^2\phi=-4\pi\rho$). Thus we can immediately write down the solution for the required gauge transformation function λ:

$$\lambda(\vec r)=\frac{1}{4\pi}\int\frac{\vec\nabla'\cdot\widetilde{\vec A}(\vec r\,')}{|\vec r-\vec r\,'|}\,d^3\vec r\,'\,. \tag{8.7}$$

We can now proceed under the assumption that $\vec{A}$ is in Coulomb gauge. Substituting into the second equation in (8.1), we find

$$\begin{aligned}\frac{4\pi}{c}\,\vec{J} &= \vec{\nabla}\times\vec{B} = \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}\,,\\ &= -\nabla^2\vec{A}\,. \end{aligned}\tag{8.8}$$

This equation also can be solved immediately; it is just a vector-valued version of the equation $\nabla^2\phi = -4\pi\rho$ from electrostatics. Thus we have

$$\vec{A}(\vec{r}) = \frac{1}{c}\int \frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\,d^3\vec{r}\,'\,. \tag{8.9}$$

We may easily calculate the magnetic field $\vec{B}$ from (8.9), by taking the curl. It is useful to note that for any vector $\vec{V}$ and any scalar f, we have the identity

$$\vec{\nabla}\times(f\,\vec{V}) = (\vec{\nabla}f)\times\vec{V} + f\,\vec{\nabla}\times\vec{V}\,. \tag{8.10}$$

Using this (and bearing in mind that $\vec{\nabla}$ acts only on functions of $\vec{r}$, and not on functions of $\vec{r}\,'$), we see from (8.9) that

$$\vec{B}(\vec{r}) = -\frac{1}{c}\int \vec{J}(\vec{r}\,')\times\vec{\nabla}\Big(\frac{1}{|\vec{r}-\vec{r}\,'|}\Big)\,d^3\vec{r}\,'\,, \tag{8.11}$$

and hence

$$\vec{B}(\vec{r}) = \frac{1}{c}\int \vec{J}(\vec{r}\,')\times\frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\,d^3\vec{r}\,'\,, \tag{8.12}$$

Suppose we consider an infinitesimal current element $I d\vec{\ell}$, where I is the current and $d\vec{\ell}$ is the infinitesimal element of length pointing along the direction of the current flow. For simplicity, let us choose the coordinate system so that this current element is located at the origin. The current density $\vec{J}(\vec{r})$ describing the current element can therefore be written as

$$\vec{J}(\vec{r}) = I\delta^2_\perp(\vec{r})\,d\vec{\ell}\,, \tag{8.13}$$

where $\delta^2_\perp(\vec{r})$ denotes the two-dimensional delta function in the plane orthogonal to $d\vec{\ell}$. This delta function describes the way in which the current density is zero everywhere except along the infinitesimal line element. For example, if the element $d\vec{\ell}$ lies along the z direction, then

$$J_x = 0\,,\qquad J_y = 0\,,\qquad J_z = I\delta(x)\delta(y)dz\,. \tag{8.14}$$

Substituting (8.13) into (8.12), we can perform the two integrations in the plane orthogonal to $d\vec{\ell}$, and thus obtain the infinitesimal contribution

$$d\vec{B}(\vec{r}) = \frac{I\,(d\vec{\ell}\times\vec{r})}{c\,r^3}\,. \tag{8.15}$$

This result is known as the Biot-Savat law, and it can be used, by integrating along a loop, to calculate the magnetic field resulting from current flow around the loop. (Historically, of course, the Biot-Savat law came first, and (8.12) was deduced from it.)

One way in which a current loop can arise is if a particle of electric charge q moves with velocity $\vec{v}$. Suppose that at a given instant the moving particle is located at the origin. It then follows from (8.15) that it will give rise to a magnetic field

$$\vec{B}(\vec{r}) = \frac{q\,\vec{v}\times\vec{r}}{c\,r^3}\,. \tag{8.16}$$

(This result assumes that the velocity $\vec{v}$ is small compared with the speed of light, so that relativistic effects can be neglected.)

Suppose that a current element $I_1\,d\vec{\ell}_1$ is placed in a magnetic field $\vec{B}$. Experiments by Ampère in the 19'th century established that it would experience a force $d\vec{F}$ given by

$$d\vec{F} = \frac{I_1}{c}\,d\vec{\ell}_1\times\vec{B}\,. \tag{8.17}$$

If the magnetic field is itself due to an infinitesimal current element $I_2\,d\vec{\ell}_2$, then from (8.15), the (doubly infinitesimal) force $d\vec{F}_{12}$ experienced by the first current element will be given by

$$d\vec{F}_{12} = \frac{I_1\,I_2}{c^2}\,\frac{d\vec{\ell}_1\times(d\vec{\ell}_2\times\vec{r}_{12})}{r_{12}^3}\,, \tag{8.18}$$

where $\vec{r}_{12}$ is the vector from $d\vec{\ell}_2$ to $d\vec{\ell}_1$. In other words, $\vec{r}_{12}=\vec{r}_2-\vec{r}_1$. Note that $d\vec{\ell}_2$ is the same thing as $d\vec{r}_2$.

The expression (8.18) can be integrated up around the two current loops, to give

$$\vec{F}_{12} = \frac{I_1\,I_2}{c^2}\oint\oint\frac{d\vec{\ell}_1\times(d\vec{\ell}_2\times\vec{r}_{12})}{r_{12}^3}\,, \tag{8.19}$$

As it stands, this expression is not manifestly (anti)symmetric under the exchange of the rôles of the two current loops. However, this can be made manifest as follows:

Using the standard identity for the vector triple product, we can write

$$d\vec{\ell}_1\times(d\vec{\ell}_2\times\vec{r}_{12}) = d\vec{\ell}_2\,(d\vec{\ell}_1\cdot\vec{r}_{12}) - (d\vec{\ell}_1\cdot d\vec{\ell}_2)\,\vec{r}_{12}\,, \tag{8.20}$$

and so (8.18) can be written as

$$d\vec{F}_{12} = \frac{I_1\,I_2}{c^2}\,d\vec{\ell}_2\Big(\frac{d\vec{\ell}_1\cdot\vec{r}_{12}}{r_{12}^3}\Big) - \frac{I_1\,I_2}{c^2}\,(d\vec{\ell}_1\cdot d\vec{\ell}_2)\,\frac{\vec{r}_{12}}{r_{12}^3}\,. \tag{8.21}$$

If we consider just the Loop 1 integration for now, the first term in (8.21) can be written as the exact differential

$$\frac{I_1\,I_2}{c^2}\,d\vec{\ell}_2\,d\Big(\frac{1}{r_{12}}\Big)\,. \tag{8.22}$$

This follows from the fact that $r_{12}^2 = (\vec{r}_2 - \vec{r}_1) \cdot (\vec{r}_2 - \vec{r}_1)$, and so (with $\vec{r}_2$ held fixed),

$$2r_{12}\, dr_{12} = -2d\vec{r}_1 \cdot (\vec{r}_2 - \vec{r}_1) = -2d\vec{\ell}_1 \cdot \vec{r}_{12}\,. \tag{8.23}$$

Thus if we first integrate (8.21) around Loop 1, the first term gives zero (since $\oint df = 0$ whenever any exact differential df is integrated around a *closed* loop). Thus we arrive at the alternative expression for the total force,

$$\vec{F}_{12} = -\frac{I_1\, I_2}{c^2} \oint\oint \frac{(d\ell_1 \cdot d\ell_2)\, \vec{r}_{12}}{r_{12}^3}\,. \tag{8.24}$$

This makes manifest the total symmetry between the rôles of the two loops.

The expression (8.17) for the force on a current element $I\, d\vec{\ell}$ in a magnetic field $\vec{B}$ can be generalised immediately to the situation where there is a current density $\vec{J}$ in an external $\vec{B}$ field. The infinitesimal force on the current density in the volume element $d^3\vec{r}$ will be given by

$$d\vec{F} = \frac{1}{c}\, \vec{J}\,(\vec{r}) \times \vec{B}(\vec{r})\, d^3\vec{r}\,, \tag{8.25}$$

and so the total force on the current distribution will be given by

$$\vec{F} = \frac{1}{c} \int \vec{J}\,(\vec{r}) \wedge \vec{B}(\vec{r})\, d^3\vec{r}\,. \tag{8.26}$$

It also follows from (8.25) that the infinitesimal torque on the element $d^3\vec{r}$ will be

$$d\vec{N} = \vec{r} \times d\vec{F} = \frac{1}{c}\, \vec{r} \times \Big(\vec{J}\,(\vec{r}) \times \vec{B}(\vec{r})\Big)\, d^3\vec{r}\,, \tag{8.27}$$

and so the total torque (measured relative to the origin) is

$$\vec{N} = \frac{1}{c} \int \vec{r} \times \Big(\vec{J}\,(\vec{r}) \times \vec{B}(\vec{r})\Big)\, d^3\vec{r}\,. \tag{8.28}$$

A further application of the expression (8.17) for the force on a current element in an external magnetic field is to the situation where a particle of charge q is moving with velocity $\vec{v}$ in the field $\vec{B}$. It then follows that it will experience a force given by

$$\vec{F} = \frac{q}{c}\, \vec{v} \times \vec{B}\,. \tag{8.29}$$

This is known as the *Lorentz force.* (We are assuming here that the velocity $\vec{v}$ is small in comparison to the speed of light.)

Finally, in this section, we note that the Maxwell equation $\vec{\nabla} \times \vec{B} = (4\pi/c)\, \vec{J}$ can be integrated to give the result known as *Ampère's Law.* Thus, integrating over an open surface Σ, with closed 1-dimensional boundary C, we have, by using Stokes' theorem

$$\int_\Sigma (\vec{\nabla} \times \vec{B}) \cdot d\vec{S} = \oint_C \vec{B} \cdot d\vec{\ell}\,, \tag{8.30}$$

the result that

$$\oint_C \vec{B} \cdot d\vec{\ell} = \frac{4\pi}{c} \int_\Sigma \vec{J} \cdot d\vec{S} \,. \tag{8.31}$$

The integral $\vec{J} \cdot d\vec{S}$ on the right-hand side is equal to the total current I passing through the area bounded by the loop C, and so we obtain Ampère's law

$$\oint_C \vec{B} \cdot d\vec{\ell} = \frac{4\pi}{c} I \,. \tag{8.32}$$

(Of course historically, Ampère's law was discovered (empirically) first, and only later was it re-expressed in the differential form $\vec{\nabla} \times \vec{B} = (4\pi/c)\,\vec{J}$.)

8.2 Magnetic field of a circular current loop

A relatively simple, and highly symmetrical, illustration of some of the methods of the previous section is provided by considering a circular conducting loop, of radius a, around which a current I is passing. We may take the loop to be centred on the origin, lying in the (x, y) plane. Using spherical polar coordinates, it is described by a current density $\vec{J}$ whose spherical-polar components J_r and J_θ vanish, whilst

$$J_\varphi(r', \theta', \varphi') = \frac{I}{a}\, \delta(\cos\theta')\, \delta(r' - a) \,. \tag{8.33}$$

It is actually helpful to re-express the current density in terms of its Cartesian coordinate components, the reason being that only in this case can we straightforwardly apply the equation (8.9) giving the vector potential $\vec{A}$ as an integral involving $\vec{J}$. Thus we shall have

$$J_x = -J_\varphi \sin\varphi' \,, \qquad J_y = J_\varphi \cos\varphi' \,, \qquad J_z = 0 \,. \tag{8.34}$$

Clearly, the system is azimuthally symmetric, and so we can, without loss of generality, take the observation point to be at $\varphi = 0$ in order to simplify the calculation. In other words, we take the observation point to lie in the (x, z) plane. This means that only the y component of $\vec{A}$ will be non-vanishing. This is the direction tangent to the circles of increasing φ, and so, in view of the azimuthal symmetry, it means that A_y in the (x, z) plane is equal to A_φ. Thus, from (8.9) and (8.33) we shall have

$$A_\varphi(r, \theta) = A_y\Big|_{\varphi=0} = \frac{I}{ac} \int dr' d\Omega' \frac{r'^2 \cos\varphi'\, \delta(\cos\theta')\, \delta(r' - a)}{|\vec{r} - \vec{r}\,'|} \,, \tag{8.35}$$

where $d\Omega' = \sin\theta'\, d\theta'\, d\varphi'$. With

$$\vec{r} = (r\sin\theta, 0, r\cos\theta) \,, \qquad \vec{r}\,' = (r'\sin\theta'\cos\varphi', r'\sin\theta'\sin\varphi', r'\cos\theta') \,, \tag{8.36}$$

we have $|\vec{r}-\vec{r}\,'|^2 = r^2 + r'^2 - 2rr'\,(\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\varphi')$, and so, after performing the integrations over the delta functions, we have

$$A_\varphi(r,\theta) = \frac{Ia}{c}\int_0^{2\pi} \frac{\cos\varphi'\, d\varphi'}{(a^2+r^2-2ar\sin\theta\cos\varphi')^{1/2}}\,. \tag{8.37}$$

The integral in (8.37) can actually be performed explicitly, although the result involves the complete elliptic integrals $K(k)$ and $E(k)$:

$$A_\varphi(r,\theta) = \frac{4Ia}{\sqrt{a^2+r^2+2ar\sin\theta}}\,\left(\frac{(2-k^2)K(k^2)-2E(k^2)}{k^2}\right)\,, \tag{8.38}$$

where

$$k^2 \equiv \frac{4ar\sin\theta}{a^2+r^2+2ar\sin\theta}\,, \tag{8.39}$$

and the complete elliptic integrals are defined by

$$K(z) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-z\sin^2\theta}}\,,\qquad E(z) = \int_0^{\pi/2} \sqrt{1-z\sin^2\theta}\,d\theta\,. \tag{8.40}$$

The spherical-polar components of the $\vec{B}$ field are then given by

$$B_r = \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\,A_\varphi)}{\partial\theta}\,,\qquad B_\theta = -\frac{1}{r}\frac{\partial(rA_\varphi)}{\partial r}\,,\qquad B_\varphi = 0\,. \tag{8.41}$$

If z is small, the elliptic integrals (8.40) admit power-series expansions

$$\begin{aligned} K(z) &= \frac{\pi}{2}\left(1+\frac{1}{4}z+\frac{9}{64}z^2+\frac{25}{256}z^3+\frac{1225}{16384}z^4+\cdots\right)\,,\\ E(z) &= \frac{\pi}{2}\left(1-\frac{1}{4}z-\frac{3}{64}z^2-\frac{5}{256}z^3-\frac{175}{16384}z^4+\cdots\right)\,, \end{aligned} \tag{8.42}$$

implying that

$$\frac{(2-k^2)K(k^2)-2E(k^2)}{k^2} = \frac{\pi k^2}{16}\left(1+\frac{3}{4}k^2+\frac{75}{128}k^4+\cdots\right)\,. \tag{8.43}$$

We see from (8.39) that k will be small if

$$r << a\,,\qquad \text{or}\qquad r >> a\,,\qquad \text{or}\qquad |\theta| << 1\,. \tag{8.44}$$

Keeping just the leading-order term in the expansion (8.43), we see that A_φ in (8.38) is given approximately by

$$A_\varphi(r,\theta) \approx \frac{I\pi a^2}{c}\,\frac{r\sin\theta}{(a^2+r^2+2ar\sin\theta)^{3/2}}\,, \tag{8.45}$$

whenever any of the conditions (8.44) holds.

Substituting (8.45) into (8.41), we obtain the approximate expressions

$$\begin{aligned} B_r &\approx \frac{I\pi a^2}{c}\,\frac{(2a^2+2r^2+ar\sin\theta)}{(a^2+r^2+2ar\sin\theta)^{5/2}}\,\cos\theta\,,\\ B_\theta &\approx -\frac{I\pi a^2}{c}\,\frac{(2a^2-r^2+ar\sin\theta)}{(a^2+r^2+2ar\sin\theta)^{5/2}}\,\sin\theta\,, \end{aligned} \tag{8.46}$$

whenever (8.44) holds.

The leading-order behaviour of the fields at large r can be read off from (8.45) and (8.46), giving

$$A_\varphi \approx \frac{I\pi a^2}{c}\,\frac{\sin\theta}{r^2}\,, \tag{8.47}$$

and

$$B_r \approx \frac{I\pi a^2}{c}\,\frac{2\cos\theta}{r^3}\,,\qquad B_\theta \approx \frac{I\pi a^2}{c}\,\frac{\sin\theta}{r^3}\,. \tag{8.48}$$

It is interesting to compare the $\vec{B}$ field of a current loop with the $\vec{E}$ field due to an electric dipole. Recall from (6.31) that the potential due to an electric dipole $\vec{p}$ at the origin is given by

$$\phi = \frac{\vec{p}\cdot\vec{r}}{r^3}\,, \tag{8.49}$$

and so if it is parallel to the z axis we shall have

$$\phi(r,\theta) = \frac{p\cos\theta}{r^2}\,. \tag{8.50}$$

This implies that the spherical-polar components of $\vec{E}$ will be given by

$$\begin{aligned} E_r &= -\frac{\partial\phi}{\partial r} = \frac{2p\cos\theta}{r^3}\,,\\ E_\theta &= -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{p\sin\theta}{r^3}\,,\\ E_\varphi &= -\frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi} = 0\,. \end{aligned} \tag{8.51}$$

Comparison with (8.48) shows that the magnetic field of a current loop, seen from afar, is dipole in character. We also see that it is natural to define the *magnetic dipole moment* m for the current loop by

$$m = \frac{I\pi a^2}{c}\,. \tag{8.52}$$

Although we were able to obtain the exact expression (8.38) for the vector potential due to a circular current loop, we ended up looking at a power-series expansion in order to understand the nature of the solution. We could also have performed a series expansion prior to evaluating the integral (8.37), thereby avoiding the need to introduce the elliptic integrals. One way of doing this is to go back to the expression (8.35), and then to substitute in the expansion (4.136), which we reproduce here:

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \sum_{\ell\ge 0}\sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1}\,\frac{r_<^\ell}{r_>^{\ell+1}}\,\bar{Y}_{\ell m}(r',\theta',\varphi')\,Y_{\ell m}(\theta,\varphi)\,. \tag{8.53}$$

Writing $\cos\varphi' = \Re(e^{i\varphi'})$, where $\Re$ denotes the real part, and recalling that $Y_{\ell m}(\theta',\varphi')$ has φ' dependence of the form $e^{im\varphi'}$ (see (4.101)), we can perform the φ' integration, as well as the r' and θ' integrations, to give

$$A_\varphi = \frac{8I\pi^2 a^2}{c}\,\Re\left(\sum_{\ell\ge 1}\frac{Y_{\ell,1}(\frac12\pi,0)}{2\ell+1}\,\frac{r_<^\ell}{r_>^{\ell+1}}\,Y_{\ell,1}(\theta,0)\right)\,, \tag{8.54}$$

where $r_<$ and $r_>$ denotes the lesser, and greater, of a and r.

Note that the $Y_{\ell,1}(\frac12\pi,0)$ are just ℓ-dependent constants that can be read off the definition (4.101):

$$Y_{\ell,1}(\tfrac12\pi,0) = \sqrt{\frac{2\ell+1}{4\pi\ell(\ell+1)}}\,P_\ell^1(0)\,. \tag{8.55}$$

Thus $Y_{\ell,1}(\frac12\pi,0)=0$ when ℓ is even, and when $\ell=2n+1$,

$$Y_{2n+1,1}(\tfrac12\pi,0) = \sqrt{\frac{2\ell+1}{4\pi\ell(\ell+1)}}\,\left[\frac{(-1)^{n+1}\Gamma(n+\frac32)}{\Gamma(n+1)\Gamma(\frac32)}\right]\,. \tag{8.56}$$

The gauge potential for the circular current loop can therefore be written as

$$A_\varphi = -\frac{\pi I a}{c}\sum_{n\ge 0}\frac{(-1)^n(2n-1)!!}{2^n\,(n+1)!}\,\frac{r_<^{2n+1}}{r_>^{2n+2}}\,P_{2n+1}^1(\cos\theta)\,, \tag{8.57}$$

where

$$(2n-1)!! \equiv (2n-1)(2n-3)(2n-5)\times\cdots\times 5\times 3\times 1\,, \tag{8.58}$$

and by definition $(2n-1)!!$ is equal to 1 when $n=0$.

The fact that the associated Legendre functions $P_\ell^m(\cos\theta)$ with $m=1$ enter here in the expansion for A_φ is a reflection of the vectorlike nature of the potential $\vec{A}$.

In order to calculate the component B_r of the $\vec{B}$ field, we see from (8.41) that it is necessary to evaluate the derivative of the associated Legendre functions $P_\ell^1(x)$ with respect to x. Recalling from (4.88) that

$$P_\ell^m(x) \equiv (-1)^m\,(1-x^2)^{m/2}\,\frac{d^m}{dx^m}\,P_\ell(x)\,, \tag{8.59}$$

we see that

$$\frac{d}{dx}\Big((1-x^2)^{1/2}P_\ell^1(x)\Big) = -\frac{d}{dx}\Big((1-x^2)\frac{dP_\ell(x)}{dx}\Big) = \ell(\ell+1)P_\ell(x)\,, \tag{8.60}$$

since we know that the Legendre polynomials $P_\ell(x)$ satisfy the standard Legendre equation $((1-x^2)P_\ell')' + \ell(\ell+1)P_\ell = 0$.

The upshot from this is that the radial component of the $\vec{B}$ field for the circular current loop is given by

$$B_r = \frac{2\pi I a}{cr} \sum_{n\ge 0} \frac{(-1)^n (2n+1)!!}{2^n\, n!}\, \frac{r_<^{2n+1}}{r_>^{2n+2}}\, P_{2n+1}(\cos\theta)\,. \tag{8.61}$$

The θ component of $\vec{B}$ is given by

$$B_\theta = \frac{2\pi I}{ca} \sum_{n\ge 0} \frac{(-1)^n (2n-1)!!}{2^n\, n!} \Big(\frac{r}{a}\Big)^{2n}\, P^1_{2n+1}(\cos\theta)\,, \quad \text{when} \quad r < a\,, \tag{8.62}$$

or by

$$B_\theta = -\frac{\pi I a^2}{cr^3} \sum_{n\ge 0} \frac{(-1)^n (2n+1)!!}{2^n\, (n+1)!} \Big(\frac{a}{r}\Big)^{2n}\, P^1_{2n+1}(\cos\theta)\,, \quad \text{when} \quad r > a\,, \tag{8.63}$$

and the φ component is zero.

8.3 Localised Current Distribution

Suppose now we consider a localised region of space within which currents are flowing, described by the current density $\vec{J}(\vec{r})$. Outside this region, it is assumed that $\vec{J}(\vec{r}) = 0$. We can proceed in a manner that is precisely analogous to earlier discussion we gave of localised charge distributions, using the expression

$$\vec{A}(\vec{r}) = \frac{1}{c} \int \frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\, d^3\vec{r}\,' \tag{8.64}$$

for the magnetic vector potential, and then Taylor expanding $|\vec{r}-\vec{r}\,'|^{-1}$ in inverse powers of r, using (6.20). Thus, keeping just the first couple of orders in the expansion, we shall have

$$\begin{aligned} \frac{1}{|\vec{r}-\vec{r}\,'|} &= \frac{1}{r} - x_i'\,\partial_i \frac{1}{r} + \cdots\,, \\ &= \frac{1}{r} + \frac{\vec{r}\,'\cdot\vec{r}}{r^3} + \cdots\,. \end{aligned} \tag{8.65}$$

It follows from (8.64) that we shall have

$$A_i(\vec{r}) = \frac{1}{cr} \int \vec{J}_i(\vec{r}\,') d^3\vec{r}\,' + \frac{x_j}{cr^3} \int x_j'\, J_i(\vec{r}\,')\, d^3\vec{r}\,' + \cdots\,. \tag{8.66}$$

The first term in (8.66) vanishes, as can be seen from the following argument. We know that the current density is conserved, $\vec{\nabla}\cdot\vec{J} = 0$. Now consider the quantity $\partial_i(x_j\, J_i)$, which is therefore given by

$$\begin{aligned}\partial_i(x_j\,J_i) &= (\partial_i x_j)\,J_i + x_j\,\partial_i J_i\,,\\ &= \delta_{ij}\,J_i\,,\\ &= J_j\,.\end{aligned} \tag{8.67}$$

If we integrate $\partial_i(x_j\,J_i)$ over all space it will give zero by the divergence theorem, since $\vec{J}$ vanishes outside some bounded domain:

$$\int \partial_i(x_j\,J_i)\,d^3\vec{r} = \int_{(\text{sphere at }\infty)} (x_j\,J_i\,d\Sigma_i = 0\,, \tag{8.68}$$

and hence we conclude that $\int J_j\,d^3\vec{r} = 0$.

To discuss the second term in (8.66), it is useful first to review the way in which one describes the vector product using index notation. This is done by introducing the totally antisymmetric tensor ϵ_{ijk}, which is defined by

$$\epsilon_{123} = 1\,. \tag{8.69}$$

The total antisymmetry under the exchange of any pair of indices implies that we must therefore have

$$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = +1\,, \qquad \epsilon_{132} = \epsilon_{321} = \epsilon_{213} = -1\,, \tag{8.70}$$

with all other components vanishing. Using this, we can clearly write the vector product $\vec{V} = \vec{A}\times\vec{B}$ as

$$V_i = \epsilon_{ijk}\,A_j\,B_k\,. \tag{8.71}$$

It is straightforward to show, by enumerating all the possible assignments for the indices,[21] that

$$\epsilon_{ijm}\,\epsilon_{k\ell m} = \delta_{ik}\,\delta_{j\ell} - \delta_{i\ell}\,\delta_{jk}\,. \tag{8.72}$$

Using this, many vector identities involving a pair of vector products can easily be proven. For example, if $\vec{V} \equiv \vec{A}\times(\vec{B}\times\vec{C})$, then we shall have

$$\begin{aligned}V_i &= \epsilon_{ijm}\,A_j\,(\vec{B}\times\vec{C})_m\,,\\ &= \epsilon_{ijm}\,A_j\,\epsilon_{mk\ell}B_k\,C_\ell\,,\\ &= \epsilon_{ijm}\,\epsilon_{k\ell m}\,A_j\,B_k\,C_\ell\,,\\ &= (\delta_{ik}\,\delta_{j\ell} - \delta_{i\ell}\,\delta_{jk})\,A_j\,B_k\,C_\ell\,,\\ &= B_i\,A_j\,C_j - C_i\,A_j\,B_j\,,\end{aligned} \tag{8.73}$$

[21]The labour involved in this proof is hugely reduced, becoming almost a triviality, if one takes note of the symmetries of the problem!

or, in 3-vector notation, the well-known identity[22]

$$\vec{V} \equiv \vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}\,(\vec{A}\cdot\vec{C}) - \vec{C}\,(\vec{A}\cdot\vec{B})\,. \tag{8.74}$$

Returning to the problem in hand we note, since $\partial_i J_i = 0$, that

$$\partial_i(x_j\, x_k\, J_i) = x_k\, J_j + x_j\, J_k\,. \tag{8.75}$$

Using the divergence theorem, the integral of the left-hand side over all space is zero, and so we conclude that

$$\int x_j\, J_k\, d^3\vec{r} = -\int x_k\, J_j\, d^3\vec{r}\,. \tag{8.76}$$

This means that the integral in the second term in (8.66) can be written as

$$\int x'_j\, J_i(\vec{r}\,')\, d^3\vec{r}\,' = -\tfrac{1}{2}\int (x'_i\, J_j - x'_j\, J_i)\, d^3\vec{r}\,'\,. \tag{8.77}$$

Now, as can easily be seen using (8.72),

$$\tfrac{1}{2}(x'_i\, J_j - x'_j\, J_i) = \tfrac{1}{2}\epsilon_{ijk}\,(\vec{r}\,'\times\vec{J})_k\,, \tag{8.78}$$

and so we conclude that the second term in (8.66) can be written as

$$\vec{A}(\vec{r}) = \frac{\vec{m}\times\vec{r}}{r^3}\,, \tag{8.79}$$

where we have defined the magnetic moment $\vec{m}$

$$\vec{m} = \frac{1}{2c}\int \vec{r}\,'\times\vec{J}(\vec{r}\,')\, d^3\vec{r}\,' \tag{8.80}$$

of the current distribution $\vec{J}$. Prior to performing the volume integration, we may define the magnetisation $\vec{\mathcal{M}}$ by

$$\vec{\mathcal{M}} = \frac{1}{2c}\,\vec{r}\times\vec{J}\,. \tag{8.81}$$

We have shown that (8.79) is the leading-order term in the multipole expansion describing the magnetic vector potential $\vec{A}$ of a localised current distribution. Let us now calculate the magnetic field $\vec{B} = \vec{\nabla}\times\vec{A}$. In index notation we have

$$\begin{aligned} B_i &= \epsilon_{ijk}\,\partial_j A_k = \epsilon_{ijk}\,\epsilon_{k\ell m}\,\partial_j\Big(\frac{m_\ell\, x_m}{r^3}\Big)\,, \\ &= (\delta_{i\ell}\,\delta_{jm} - \delta_{im}\,\delta_{j\ell})\,\partial_j\Big(\frac{m_\ell\, x_m}{r^3}\Big)\,, \\ &= m_i\,\partial_j\Big(\frac{x_j}{r^3}\Big) - m_j\,\partial_j\Big(\frac{x_i}{r^3}\Big)\,. \end{aligned} \tag{8.82}$$

[22]Remembering just the simple expression (8.72) enables one to prove easily, "on demand," almost all the identities of three-dimensional vector calculus.

Now, away from the origin (i.e. for $r > 0$) we have

$$\partial_j\Big(\frac{x_j}{r^3}\Big) = \frac{\partial_j x_j}{r^3} - \frac{3x_j}{r^4}\,\frac{x_j}{r} = \frac{3}{r^3} - \frac{3}{r^3} = 0\,, \tag{8.83}$$

and

$$\partial_j\Big(\frac{x_i}{r^3}\Big) = \frac{\delta_{ij}}{r^3} - \frac{3x_i\,x_j}{r^5}\,, \tag{8.84}$$

and hence we find

$$\vec{B} = \frac{3(\vec{m}\cdot\vec{n})\,\vec{n} - \vec{m}}{r^3}\,, \qquad \text{for} \qquad r > 0\,, \tag{8.85}$$

where as usual we define the unit vector $\vec{n} = \vec{r}/r$. Thus $\vec{B}$ has exactly the same form as the electric field of an electric dipole $\vec{p}$ (see (6.36)).

It is of interest also to consider the expression for $\vec{B}$ including the origin $r = 0$.[23] Recalling that $\nabla^2(1/r) = -4\pi\,\delta^3(\vec{r})$, we see that

$$\partial_j\Big(\frac{x_j}{r^3}\Big) = 4\pi\,\delta^3(\vec{r})\,. \tag{8.86}$$

What about the other term in the last line of (8.82)? It is clear that $\partial_j(x_i/r^3)$ must also have a delta-function term at the origin, since we already know that taking its trace gives $\partial_j(x_j/r^3) = 4\pi\,\delta^3(\vec{r})$. Obviously no direction in 3-dimensional space can be preferred over any other, and so the delta-function term in $\partial_j(x_i/r^3)$ must be *isotropic*, i.e. proportional to δ_{ij}. To be consistent with the trace formula, it follows that the full expression for $\partial_j(x_i/r^3)$ must be the $r > 0$ result given in (8.84) plus a delta function term, giving in total

$$\partial_j\Big(\frac{x_i}{r^3}\Big) = \frac{\delta_{ij}}{r^3} - \frac{3x_i\,x_j}{r^5} + \frac{4\pi}{3}\delta_{ij}\,\delta^3(\vec{r})\,, \tag{8.87}$$

Putting all the above results together, it follows from (8.82) that

$$\vec{B} = \frac{3(\vec{m}\cdot\vec{n})\,\vec{n} - \vec{m}}{r^3} + \frac{8\pi}{3}\,\vec{m}\,\delta(\vec{r})\,. \tag{8.88}$$

If the current distribution takes the form of a planar closed loop of current (i.e. a current flowing round a planar wire loop), then the general expression (8.80) for the magnetic moment reduces to

$$\vec{m} = \frac{I}{2c}\oint \vec{r}\times d\vec{r}, \tag{8.89}$$

where I is the current. Note that the magnetic moment is perpendicular to the plane of the loop. Since $\frac{1}{2}\vec{r}\times d\vec{r}$ is the area element of the triangular wedge whose vertices lie at the origin, and the points $\vec{r}$ and $\vec{r}+d\vec{r}$ on the loop, it follows that $\frac{1}{2}\oint\vec{r}\times d\vec{r}$ gives the area

[23]Note that it only really makes sense to do this in the case of a "point dipole" of zero size.

of the loop, and so the magnitude m of the magnetic moment for a planar current loop of area A is just given by

$$m = \frac{IA}{c}\,. \tag{8.90}$$

This result generalises, to a planar loop of arbitray shape, the result obtained in (8.52) in the case of a circular current loop.

Another special case we may consider is when the current distribution is generated by N point charges q_a, for $1 \le a \le N$, located at points $\vec r_a$ and moving with velocities $\vec v_a = d\vec r_a/dt$. The current density is therefore given by

$$\vec J(\vec r) = \sum_{a=1}^{N} q_a\, \vec v_a\, \delta^3(\vec r - \vec r_a)\,. \tag{8.91}$$

Inserting this into (8.80), the integration can be performed, giving

$$\vec m = \frac{1}{2c} \sum_{a=1}^{N} q_a\, (\vec r_a \times \vec v_a)\,. \tag{8.92}$$

Now, if the a'th particle has mas M_a, then its orbital angular momentum is given by

$$\vec L_a = M_a\, (\vec r_a \times \vec v_a)\,, \tag{8.93}$$

and so we have

$$\vec m = \sum_{a=1}^{N} \frac{q_a}{2cM_a}\, \vec L_a\,. \tag{8.94}$$

If all the particles have the same charge to mass ratio,

$$\frac{q_a}{M_a} = \frac{q}{M}\,, \qquad \text{for each } a\,, \tag{8.95}$$

then we have the simple relation

$$\vec m = \frac{q}{2Mc}\, \vec L\,, \tag{8.96}$$

where

$$\vec L = \sum_{a=1}^{N} \vec L_a \tag{8.97}$$

is the total orbital angular momentum of the system of particles.

8.4 Force on a current distribution in an external $\vec B$ field

Suppose a localised current distribution is located in a region where there is an externally-generated magnetic field $\vec B(\vec r)$ (which may be position dependent). If we assume that the magnetic field varies slowly with position, then we can make a Taylor expansion of $\vec B(\vec r)$

around some point (which can conveniently be taken to be the origin), and keep just the leading-order terms. Thus we shall have

$$B_i(\vec{r}) = B_i(0) + \vec{r}\cdot\vec{\nabla}B_i(0) + \cdots\,, \tag{8.98}$$

where, of course, in the second term the argument is set to zero *after* taking the gradient. Substituting into the expression

$$\vec{F} = \frac{1}{c}\int \vec{J}(\vec{r})\wedge\vec{B}(\vec{r})\,d^3\vec{r} \tag{8.99}$$

for the force on a current distribution, we therefore find

$$\vec{F} = -\frac{1}{c}\,\vec{B}(0)\times\int\vec{J}(\vec{r}\,')\,d^3\vec{r}\,' + \frac{1}{c}\int\vec{J}(\vec{r}\,')\times[(\vec{r}\,'\cdot\vec{\nabla})\vec{B}(0)]\,d^3\vec{r}\,' + \cdots\,. \tag{8.100}$$

As we already saw earlier, the integral in the first term vanishes, and so the leading-order contribution to the force comes from the second term. In index notation, the second term is

$$F_i = \frac{1}{c}\,\epsilon_{ijk}\,(\partial_\ell B_k)(0)\int x'_\ell\,J_j(\vec{r}\,')\,d^3\vec{r}\,'\,. \tag{8.101}$$

Using (8.77), (8.78) and (8.80), we therefore have

$$\begin{aligned} F_i &= -\epsilon_{ijk}\,\epsilon_{j\ell m}\,m_m\,\partial_\ell B_k(0)\,,\\ &= -m_i\,\partial_k B_k(0) + m_k\,\partial_i B_k(0)\,,\\ &= m_k\,\partial_i B_k(0)\,, \end{aligned} \tag{8.102}$$

where, in getting to the final line, we have used $\vec{\nabla}\cdot\vec{B} = 0$. Thus, since $\vec{m}$ is a constant, we can write

$$\vec{F} = \vec{\nabla}(\vec{m}\cdot\vec{B})\,. \tag{8.103}$$

The expression (8.103) for the force on a magnetic dipole $\vec{m}$ in a magnetic field $\vec{B}$ shows that we can define a potential energy

$$U = -\vec{m}\cdot\vec{B}\,, \tag{8.104}$$

in terms of which the force is given by $\vec{F} = -\vec{\nabla}U$.

Note that since $\vec{B}$ is assumed to be generated by distant current sources, and so $\vec{\nabla}\times\vec{B} = 0$ in the region under consideration, we have $\epsilon_{k\ell m}\partial_\ell B_m = 0$ and hence $\epsilon_{ijk}\,\epsilon_{k\ell m}\partial_\ell B_m = 0$. Using (8.72), this implies that

$$\partial_i B_j - \partial_j B_i = 0\,. \tag{8.105}$$

Using also the constancy of $\vec{m}$, we can then manipulate (8.103) to give

$$F_i = \partial_i(m_j B_j) = m_j\,\partial_i B_j = m_j\,\partial_j B_i\,, \tag{8.106}$$

and so we can give the alternative expression

$$\vec{F} = (\vec{m}\cdot\vec{\nabla})\vec{B} \tag{8.107}$$

for the force on the magnetic moment.

To calculate the torque on the current distribution, we substitute (8.98) into the general expression for the torque that we derived previously,

$$\vec{N} = \frac{1}{c}\int \vec{r}\,' \times \Big(\vec{J}(\vec{r}\,') \times \vec{B}(\vec{r}\,')\Big)\,d^3\vec{r}\,'\,. \tag{8.108}$$

Unlike in the calculation of the force, here the first term in (8.98) gives a non-zero contribution, and so to leading order we have

$$\vec{N} = \frac{1}{c}\int \vec{r}\,' \times \Big(\vec{J}(\vec{r}\,') \times \vec{B}(0)\Big)\,d^3\vec{r}\,'\,. \tag{8.109}$$

Expanding out the vector triple product, and writing in index notation, we have

$$N_i = \frac{1}{c}\,B_j(0)\int x_j\,J_i(\vec{r})\,d^3\vec{r} - \frac{1}{c}\,B_i(0)\int x_j\,J_j(\vec{r})\,d^3\vec{r}\,. \tag{8.110}$$

(We have dropped the primes on the integration variables $\vec{r}$ since there is no need to distinguish between $\vec{r}$ and $\vec{r}\,'$ in this calculation.)

The second term in (8.110) integrates to zero. This can be seen by integrating the identity

$$\vec{\nabla}\cdot(r^2\,\vec{J}\,) = 2\vec{r}\cdot\vec{J} + r^2\,\vec{\nabla}\cdot\vec{J} = 2\vec{r}\cdot\vec{J} \tag{8.111}$$

over all space, and using the divergence theorem to turn the left-hand side into a surface integral at infinity. Using (8.77), (8.78) and (8.80), the first term in (8.110) can be seen to give

$$\vec{N} = \vec{m}\times\vec{B}(0)\,. \tag{8.112}$$

8.5 Magnetically permeable media

In section 7, we discussed the phenomenological description of dielectric media, in which one introduces a macroscopic $\vec{D}$ field in addition to the fundamental electric field $\vec{E}$. The essential idea is that for many purposes, one can give a macroscopic description of the effect of a piece of dielectric medium, such as a salt crystal or a block of glass, in which the microscopic contributions of each atom or molecule within the medium are averaged over, so that on the large scale a relatively simple description of the electrical properties of the material as a whole can be given.

In a very similar vein, in magnetostatics one may give a macroscopic description of magnetically permeable materials. As in the case of electrostatics, the idea here is that whilst one could, in principle, simply apply the fundamental Maxwell equations of magnetostatics,

$$\vec\nabla\cdot\vec B = 0\,,\qquad \vec\nabla\times\vec B = \frac{4\pi}{c}\,\vec J\,, \tag{8.113}$$

at the microscopic level, to the entire system of atoms or molecules, it instead suffices, for many purposes, to work with a macroscopically averaged description. In this description, one introduces a phenomenological field $\vec H$ in addition to the fundamental field $\vec B$ of magnetostatics. Rather confusingly, $\vec H$ is called the *magnetic field*, while the fundamental field $\vec B$ is called the *magnetic induction*.

Since the derivation of the phenomenological equations for magnetostatics is closely parallel to the derivation for electrostatics in section 7, we shall just present the final conclusions here; a derivation can be found in any of the standard textbooks.

The effect of the ensemble of atoms or molecules in the medium is to give rise to a magnetic moment density $\vec M(\vec r)$, which comes from an averaging over all the atoms or molecules. The magnetic induction $\vec B$ still satisfies the original Maxwell equation $\vec\nabla\cdot\vec B=0$, and so we may still write

$$\vec B = \vec\nabla\times\vec A\,, \tag{8.114}$$

but the equation (8.64) giving the vector potential for a localied current distribution is now modified to

$$\vec A(\vec r) = \frac{1}{c}\int \frac{\vec J(\vec r\,') + c\,\vec\nabla'\times\vec M(\vec r\,')}{|\vec r-\vec r\,'|}\, d^3\vec r\,'\,. \tag{8.115}$$

The magnetisation can thus be thought of as giving rise to an effective current density

$$\vec J_M = c\,\vec\nabla\times\vec M\,. \tag{8.116}$$

The second of the Maxwell equations in (8.113) now becomes

$$\vec\nabla\times\vec B = \frac{4\pi}{c}\,\vec J + 4\pi\,\vec\nabla\times\vec M\,. \tag{8.117}$$

We are therefore led to define a new field $\vec H$ by

$$\vec H = \vec B - 4\pi\,\vec M\,. \tag{8.118}$$

The free-space Maxwell equations (8.113) therefore become

$$\vec{\nabla} \cdot \vec{B} = 0\,, \qquad \vec{\nabla} \times \vec{H} = \frac{4\pi}{c}\,\vec{J} \tag{8.119}$$

in the presence of a magnetically-permeable medium. Recall that in electrostatics, we had the Maxwell equations

$$\vec{\nabla} \cdot \vec{D} = 4\pi\,\rho\,, \qquad \vec{\nabla} \times \vec{E} = 0\,, \tag{8.120}$$

in the presence of a dielectric medium. Note that in each case it is the field equation with the source term on the right-hand side that is modified in the presence of the medium, whilst the equation that has no source term is unmodified. In other words, the phenomenological description of dielectric or permeable media modifies the *field equations* but leaves the *Bianchi identities* unchanged.

In order to apply the phenomenological description of magnetostatics, it is necessary to know the so-called *constitutive relation* between $\vec{B}$ and $\vec{H}$. This can be quite a complicated business in general, especially in the case of ferromagnetic materials where there is not even a single-valued functional relation between $\vec{B}$ and $\vec{H}$.

The simplest materials to consider are paramagnetic and diamagnetic media. For these, there is a linear relation between $\vec{B}$ and $\vec{H}$, with

$$\vec{B} = \mu\,\vec{H}\,, \tag{8.121}$$

where μ is a constant called the magnetic permeability. For paramagnetic materials, μ is slightly greater than 1, whilst for diamagnetic materials, μ is slightly less than 1. (The deviations from 1 are only a few parts in 10^5.)

For many ferromagnetic materials, a relation of the form (8.121) is approximately valid, provided the fields are sufficiently weak. In these materials, μ is typically in the range between 10 and 10^4.

In the remainder of our consideration of magnetically-permeable media, we shall assume that the simple linear constitutive relation (8.121) holds.

8.6 Boundary conditions at medium interfaces

In section (1.4) we gave a discussion of the boundary condition that must be satisfied at an interface between two magnetically-permeable media. These were obtained by considering a "pill-box" volume integration of the equation $\vec{\nabla}\cdot\vec{B} = 0$, and integrations of $\vec{\nabla}\times\vec{H} = (4\pi/c)\vec{J}$ over slender rectangular loops straddling the boundary. These lead, respectively, to the conditions

$$\vec{n}\cdot(\vec{B}_2-\vec{B}_1)=0\,,$$
$$\vec{n}\times(\vec{H}_2-\vec{H}_1)=\frac{4\pi}{c}\,\vec{K}\,, \tag{8.122}$$

where $\vec{n}$ is the unit vector normal to the interface, pointing from medium 1 into medium 2, and $\vec{K}$ is the surface current density.

The first equation in (8.122) says that the normal component of $\vec{B}$ must be continuous across the boundary. In many common situations there will be no surface currents at the interface, and then the second equation in (8.122) just says that the tangential components of $\vec{H}$ must be continuous across the boundary.

8.7 Techniques for solving boundary-value problems in magnetostatics

There are various techniques that can be applied in order to solve for the magnetic fields in a boundary-value problem in magnetostatics. Depending upon the the circumstances, one or another may be more convenient. For concreteness and simplicity, we shall typically focus on situations where the simple relation $\vec{B}=\mu\,\vec{H}$ holds, with the assumption that in any given region μ is a constant. (We shall allow μ to take different constant values in different regions, as would be the case in a typical boundary-value problem with an interface between media.)

8.7.1 Using the vector potential

The most direct way to formulate the problem is just to take the Maxell equations of magnetostatics,

$$\vec{\nabla}\cdot\vec{B}=0\,,\qquad \vec{\nabla}\times\vec{H}=\frac{4\pi}{c}\,\vec{J}, \tag{8.123}$$

solve the first equation by writing $\vec{B}$ in terms of the vector potential,

$$\vec{B}=\vec{\nabla}\times\vec{A}\,, \tag{8.124}$$

and then plug this into the second of the Maxwell equations. Of course in general, this will give a very complicated result if there is a complicated constitutive relation between $\vec{B}$ and $\vec{H}$, but if we make the assumption that $\vec{B}=\mu\,\vec{H}$, where μ is piecewise constant, then in the various regions we shall have $\vec{\nabla}\times(\vec{\nabla}\times\vec{A})=(4\pi\mu/c)\,\vec{J}$. Expanding out the vector triple product, we then obtain, by working in the Coulomb gauge $\vec{\nabla}\cdot\vec{A}=0$, the equation

$$\nabla^2\,\vec{A}=-\frac{4\pi\mu}{c}\,\vec{J}. \tag{8.125}$$

This is very similar to the equation in free space, except that now we have the permeability μ appearing on the right-hand side.

8.7.2 Magnetic scalar potential satisfying Laplace's equation

If there are no free currents in the problem, as is often the case, the Maxwell field equation becomes

$$\vec{\nabla} \times \vec{H} = 0\,, \tag{8.126}$$

and this can be solved by writing $\vec{H}$ as the gradient of a scalar, just as one does for the electric field in electrostatics. Thus we may write

$$\vec{H} = -\vec{\nabla}\Phi_M\,, \tag{8.127}$$

where Φ_M is the *magnetic vector potential.* If we assume the simple constitutive relation $\vec{B} = \mu\,\vec{H}$ between $\vec{B}$ and $\vec{H}$, where μ is piecewise constant, then in each region the remaining Maxwell equation $\vec{\nabla}\cdot\vec{B} = 0$ becomes

$$\nabla^2\Phi_M = 0\,. \tag{8.128}$$

Solving boundary-value magnetostatics problems where $\vec{J} = 0$ is thus closely analogous to solving boundary-value problems in electrostatics; one just has to solve Laplace's equation in the various regions, and then impose the boundary conditions at interfaces as discussed in section 8.6.

As an illustration of the method, consider a spherical shell of material with magnetic permeability μ, with inner radius a and outer radius b. The regions inside and outside the shell are assumed to be empty space. Outside the shell, we take the magnetic field to be asymptotically uniform, $\vec{B} \longrightarrow \vec{B}_0$, with $\vec{B}_0$ directed along the z axis. The problem is to solve for the magnetic fields everywhere. A particular point of interest will be to calculate the magnetic field inside the shell, in order to study the phenomenon of *magnetic shielding.*

In the regions $0 \le r < a$ and $r > b$ we have $\vec{B} = \vec{H}$, whilst in the region $a < r < b$ we have $\vec{B} = \mu\,\vec{H}$. There are no external currents, and so the conditions discussed above apply. There is azimuthal symmetry around the z axis, and so in spherical polar coordinates the magnetic scalar potential will depend only on r and θ. Since it must satisfy Laplace's equation in the three regions, it must take the form

$$0 < r < a: \qquad \Phi_M(r,\theta) = \sum_{\ell\ge 0} \alpha_\ell\, r^\ell\, P_\ell(\cos\theta)\,,$$

$$
\begin{aligned}
a < r < b: \qquad & \Phi_M(r,\theta) = \sum_{\ell\ge 0} \Big(\beta_\ell\, r^\ell + \frac{\gamma_\ell}{r^{\ell+1}}\Big)\, P_\ell(\cos\theta)\,,\\
r > b: \qquad & \Phi_M(r,\theta) = -B_0\, r\cos\theta + \sum_{\ell\ge 0} \frac{\delta_\ell}{r^{\ell+1}}\, P_\ell(\cos\theta)\,.
\end{aligned} \tag{8.129}
$$

The boundary conditions at $r = a$ and $r = b$ require that the radial components of $\vec B$ be continuous, and the tangential components of $\vec H$ be continuous (as in (8.122), with the surface current $\vec K = 0$). We therefore have the conditions

$$
\begin{aligned}
\mu\, \frac{\partial\Phi_M}{\partial r}\Big|_{r=a^+} &= \frac{\partial\Phi_M}{\partial r}\Big|_{r=a^-}\,, \qquad & \frac{\partial\Phi_M}{\partial\theta}\Big|_{r=a^+} &= \frac{\partial\Phi_M}{\partial\theta}\Big|_{r=a^-}\,,\\
\mu\, \frac{\partial\Phi_M}{\partial r}\Big|_{r=b^-} &= \frac{\partial\Phi_M}{\partial r}\Big|_{r=b^+}\,, \qquad & \frac{\partial\Phi_M}{\partial\theta}\Big|_{r=b^-} &= \frac{\partial\Phi_M}{\partial\theta}\Big|_{r=b^+}\,,
\end{aligned} \tag{8.130}
$$

where $a^\pm$ and $b^\pm$ mean that a and b are approached from above (plus sign) or below (minus sign) respectively. Plugging in the expansions, and using the linear independence of the Legendre polynomials (and their derivatives with respect to θ), we obtain the equations

$$
\begin{aligned}
\ell\,\mu\,\beta_\ell\, a^{\ell-1} - (\ell+1)\,\mu\,\gamma_\ell\, a^{-\ell-2} - \ell\,\alpha_\ell\, a^{\ell-1} &= 0\,,\\
\beta_\ell\, a^\ell + \gamma_\ell\, a^{-\ell-1} - \alpha_\ell\, a^\ell &= 0\,,\\
\ell\,\mu\,\beta_\ell\, b^{\ell-1} - (\ell+1)\,\mu\, b^{-\ell-2} + (\ell+1)\,\delta_\ell\, b^{-\ell-2} &= -B_0\,\delta_{\ell,1}\,,\\
\beta_\ell\, b^\ell + \gamma_\ell\, b^{-\ell-1} - \delta_\ell\, b^{-\ell-1} &= -b\,B_0\,\delta_{\ell,1}\,,
\end{aligned} \tag{8.131}
$$

where $\delta_{\ell,1}$ is a Kronecker delta which vanishes except when $\ell = 1$, in which case it equals 1. It is straightforward to solve the four equations (8.131) for the four unknowns α_ℓ, β_ℓ, γ_ℓ and δ_ℓ, for each value of ℓ. The result is

$$
\alpha_\ell = \beta_\ell = \gamma_\ell = \delta_\ell = 0\,, \quad \text{for} \quad \ell \ne 1\,, \tag{8.132}
$$

while for $\ell = 1$ we obtain

$$
\begin{aligned}
\alpha_1 &= -\frac{9\mu}{(\mu+2)(2\mu+1) - 2a^3\, b^{-3}\,(\mu-1)^2}\, B_0\,,\\
\beta_1 &= -\frac{3(2\mu+1)\, a^3}{(\mu+2)(2\mu+1) - 2a^3\, b^{-3}\,(\mu-1)^2}\, B_0\,,\\
\gamma_1 &= -\frac{3(\mu-1)\, a^3}{(\mu+2)(2\mu+1) - 2a^3\, b^{-3}\,(\mu-1)^2}\, B_0\,,\\
\delta_1 &= \frac{(2\mu+1)(\mu-1)(b^3-a^3)}{(\mu+2)(2\mu+1) - 2a^3\, b^{-3}\,(\mu-1)^2}\, B_0\,.
\end{aligned} \tag{8.133}
$$

The magnetic field inside the shell is uniform, parallel to $\vec B_0$, and has magnitude $(-\alpha_1)$. If $\mu >> 1$, as is easily possible for a high permeability material, which might have $\mu \sim 10^6$, we see that

$$
-\alpha_1 \approx \frac{9}{2\mu(1 - a^3\, b^{-3})}\, B_0\,. \tag{8.134}
$$

This shows that even with a fairly thin shell of the permeable medium, a high degree of magnetic shielding can be achieved.

Outside the shell, the magnetic field is the sum of the uniform field $\vec{B}_0$ plus a magnetic dipole term of magnitude α_1.

8.7.3 Magnetic scalar potential for hard ferromagnets

Another circumstance where we can make use of the magnetic scalar potential is in the case of a so-called *hard ferromagnet*. Such a ferromagnet is commonly also called a *permanent magnet*, namely a ferromagnetic material that has its own intrinsic magnetisation, even in the absence of externally-applied magnetic fields. In fact, to a good approximation, the magnetisation inside the hard ferromagnet is independent of any external magnetic field. Thus a common situation is to have a hard ferromagnet, with known magnetisation $\vec{M}$, immersed in free space. One might also include externally-applied magnetic fields.

Let us consider such a situations where there are no external currents, and so we have the Maxwell equations

$$\vec{\nabla}\cdot\vec{B} = 0\,, \qquad \vec{\nabla}\times\vec{H} = 0\,, \tag{8.135}$$

where, as always,

$$\vec{H} = \vec{B} - 4\pi\,\vec{M}\,. \tag{8.136}$$

We solve the second equation in (8.135) by writing

$$\vec{H} = -\vec{\nabla}\Phi_M\,. \tag{8.137}$$

The first equation becomes, upon using (8.136),

$$\vec{\nabla}\cdot\vec{H} = -4\pi\,\vec{\nabla}\cdot\vec{M}\,, \tag{8.138}$$

and hence

$$\nabla^2\Phi_M = 4\pi\,\vec{\nabla}\cdot\vec{M}\,. \tag{8.139}$$

This may be solved by writing

$$\Phi_M(\vec{r}) = -\int \frac{\vec{\nabla}'\cdot\vec{M}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\,d^3\vec{r}\,'\,. \tag{8.140}$$

The magnetisation $\vec{M}$ will be non-zero only inside the ferromagnet.

Let us consider, as an example, a uniformly magnetised sphere of ferromagnetic material in free space, with no externally-appplied magnetic fields. We shall assume that inside the sphere, of radius a, there is a uniform magnetisation $\vec{M}_0$; i.e. $\vec{M}_0$ is constant. It will be convenient presently to assume that axes are chosen so that $\vec{M}_0$ is aligned along the z direction, and that the sphere is centred on the origin.

Naively, one might think that since the magnetisation is piecewise constant in this problem (equal to $\vec{M}_0$ inside the sphere, and equal to zero outside the sphere), then $\vec{\nabla}\cdot\vec{M}$ would be zero and hence (8.140) would be zero. (A manifestly absurd conclusion, since obviously the magnet will produce a non-vanishing magnetic field outside the sphere.) The flaw in the naive argument is, of course, that the magnetisation undergoes a discontinuous jump at the surface of the sphere, and so $\vec{\nabla}\cdot\vec{M}$ will be equal to a delta-function at $r=a$.

One way to calculate (8.140) is to note, using the vector identity $\vec{\nabla}\cdot(f\,\vec{M}) = \vec{M}\cdot\vec{\nabla}f + f\,\vec{\nabla}\cdot\vec{M}$, that Φ_M can be written as

$$\Phi_M(\vec{r}) = -\int \vec{\nabla}'\cdot\Big[\frac{\vec{M}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\Big]\,d^3\vec{r}\,' + \int \vec{M}(\vec{r}\,')\cdot\vec{\nabla}'\Big[\frac{1}{|\vec{r}-\vec{r}\,'|}\Big]\,d^3\vec{r}\,'\,. \tag{8.141}$$

Using the divergence theorem, the first term can be turned into a surface integral over the sphere at infinity, and therefore it gives zero since $\vec{M}=0$ outside the ferromagnet. For the second term, we use the fact that

$$\vec{\nabla}'\Big[\frac{1}{|\vec{r}-\vec{r}\,'|}\Big] = -\vec{\nabla}\Big[\frac{1}{|\vec{r}-\vec{r}\,'|}\Big]\,, \tag{8.142}$$

and so we have

$$\Phi_M(\vec{r}) = -\vec{\nabla}\cdot\int \frac{\vec{M}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\,d^3\vec{r}\,'\,. \tag{8.143}$$

In the problem at hand, the magnetisation $\vec{M}$ is equal to the constant vector $\vec{M}_0$, parallel to the z axis, inside the sphere of radius a. Outside the sphere, we have $\vec{M}=0$. Thus (8.143) becomes

$$\Phi_M(\vec{r}) = -\vec{M}_0\cdot\vec{\nabla}\int_0^a r'^2\,dr'\int d\Omega'\frac{1}{|\vec{r}-\vec{r}\,'|}\,, \tag{8.144}$$

where $d\Omega' = \sin\theta'\,d\theta'\,d\varphi'$. Now recall from equation (4.136) that we can write

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \sum_{\ell\ge 0}\sum_{m=-\ell}^{\ell}\frac{4\pi}{2\ell+1}\,\frac{r_<^\ell}{r_>^{\ell+1}}\,\bar{Y}_{\ell m}(\theta',\varphi')\,Y_{\ell m}(\theta,\varphi)\,r.\,, \tag{8.145}$$

Since

$$\int d\Omega\,\bar{Y}_{\ell' m'}(\theta\,\varphi)\,Y_{\ell m}(\theta,\varphi) = \delta_{\ell\ell'}\,\delta_{mm'}\,, \tag{8.146}$$

and $Y_{0,0}(\theta,\varphi) = 1/\sqrt{4\pi}$, it follows that

$$\int \bar{Y}_{\ell m}(\theta',\varphi')\,d\Omega' = \sqrt{4\pi}\,\delta_{\ell,0}\,\delta_{m,0}\,. \tag{8.147}$$

Thus we have

$$\int \frac{d\Omega'}{|\vec{r}-\vec{r}\,'|} = \sum_{\ell\ge 0}\sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1}\,\frac{r_<^\ell}{r_>^{\ell+1}}\,Y_{\ell m}(\theta,\varphi)\int d\Omega'\,\bar Y_{\ell m}(\theta',\varphi')\,,$$
$$= \frac{(4\pi)^{3/2}}{r_>}\,Y_{0,0}(\theta,\varphi) = \frac{4\pi}{r_>}\,, \tag{8.148}$$

with $r_>$ being the greater of r and r'. Equation (8.144) therefore becomes

$$\Phi_M(\vec{r}) = -4\pi\,\vec{M}_0\cdot\vec{\nabla}\int_0^a \frac{r'^2\,dr'}{r_>}\,. \tag{8.149}$$

At points outside the sphere, we have $r > a$. On the other hand, the integration over r' runs only up to $r' = a$. Therefore in the exterior region we have $r_> = r$, and so (8.149) gives

$$\Phi_M(\vec{r}) = -\frac{4\pi a^3}{3}\,\vec{M}_0\cdot\vec{\nabla}\frac{1}{r}\,,\qquad (r > a)\,. \tag{8.150}$$

If, on the other hand, we look at the magnetic potential inside the sphere, i.e. for points with $0\le r < a$, then $r_>$ is equal to r for $0 \le r' < r$, while $r_>$ is equal to r' for $r < r' < a$. In this case we have

$$\int_0^a \frac{r'^2\,dr'}{r_>} = \frac{1}{r}\int_0^r r'^2\,dr' + \int_r^a r'\,dr' = -\tfrac{1}{6}r^2 + \tfrac{1}{2}a^2\,, \tag{8.151}$$

and so

$$\Phi_M(\vec{r}) = \frac{2\pi}{3}\,\vec{M}_0\cdot\vec{\nabla}r^2\,,\qquad (r < a)\,. \tag{8.152}$$

Since we are taking $\vec{M}_0$ to be parallel to the z axis, and since $\vec{\nabla}r = \vec{r}/r$, we therefore have

$$\Phi_M = \frac{4\pi a^3\,M_0}{3}\,\frac{\cos\theta}{r^2}\,,\qquad (r > a)\,,$$
$$\Phi_M = \frac{4\pi\,M_0}{3}\,r\,\cos\theta\,,\qquad (r < a)\,. \tag{8.153}$$

We see that outside the sphere, the magnetic field $\vec{B} = -\vec{\nabla}\Phi_M$ is a magnetic dipole field with magnetic moment

$$\vec{m} = \frac{4\pi a^3}{3}\,\vec{M}_0\,. \tag{8.154}$$

Inside sphere, we have $\vec{H} = -\vec{\nabla}\Phi_M = -\vec{\nabla}(4\pi\,M_0\,z/3)$ and so

$$\vec{H} = -\frac{4\pi}{3}\,\vec{M}_0\,. \tag{8.155}$$

Using (8.136), we then have

$$\vec{B} = \frac{8\pi}{3}\,\vec{M}_0\,. \tag{8.156}$$

8.7.4 Magnetised sphere in an external field

As an elaboration of the discussion above of a magnetised sphere, we can superpose on it a uniform magnetic induction $\vec{B}_0 = \vec{H}_0$ throughout all space. This means that inside the sphere, the expressions (8.155) and (8.156) will be modified to become

$$\vec{H} = \vec{B}_0 + \frac{8\pi}{3}\,\vec{M}_0\,, \qquad \vec{B} = \vec{B}_0 - \frac{4\pi}{3}\,\vec{M}_0\,. \tag{8.157}$$

One can now replace the original problem, in which the sphere was composed of a permanently magnetised ferromagnetic material, by a problem in which the sphere is instead a permeable medium of magnetic permeability μ. This means that $\vec{B}$ and $\vec{H}$ in (8.157) will now be related by $\vec{B} = \mu\,\vec{H}$. This allows us to solve for the magnetisation $\vec{M}_0$, finding

$$\vec{M}_0 = \frac{3(\mu-1)}{4\pi(\mu+2)}\,\vec{B}_0\,. \tag{8.158}$$

This result is a direct magnetic analogue of the expression (7.65) that we found when calculating the polarisation of a dielectric sphere in an asymptotically uniform electric field.