615–Maths Methods in Theoretical Physics

Problem Sheet 7

(1a) Write two quaternions \( A \) and \( B \) as ordered pairs of complex numbers, \( A = (a, b) \) and \( B = (c, d) \). Using the multiplication rule \( AB = (ac - \overline{db}, da + b\bar{c}) \), show that in general multiplication is non-commutative, i.e. \( AB \neq BA \). Show that multiplication of any three quaternions is associative; \( A(BC) = (AB)C \), where \( C = (e, f) \). With the conjugate defined by \( \overline{A} = (\overline{a}, -b) \), show that \( \overline{AA} = A\overline{A} = (a\overline{a} + b\overline{b}, 0) \), which is just the real number \( a\overline{a} + b\overline{b} \).

(1b) Define octonions \( A, B \) and \( C \) by \( A = (a, b), B = (c, d), C = (e, f) \), where \( a, b, c, d, e \) and \( f \) are quaternions. Using the multiplication rule \( AB = (ac - \overline{db}, da + b\bar{c}) \), show that multiplication is non-associative.

(1c) The conjugate is defined by \( \overline{A} = (\overline{a}, -b) \). Show that any two octonions \( A \) and \( B \) satisfy

\[ \overline{A} (AB) = (\overline{AA}) B, \]

which is needed to establish that the octonions form a division algebra, as discussed in the lectures.

(2) Prove that if one defines the derivative of \( f(z) = u(x, y) + i v(x, y) \) at \( z \) by taking the limit of \( (f(z + \delta z) - f(z))/\delta z \), where \( \delta z \) approaches zero along a line at angle \( \psi \) in the complex plane (i.e. take \( \delta z = \epsilon e^{i\psi} \) as the real constant \( \epsilon \) goes to zero), then the answer is independent of \( \psi \) if the Cauchy-Riemann equations hold.

(3.) Show that each of the functions

\begin{align*}
\text{i) } \quad u &= x^4 - 6x^2 y^2 + y^4 \\
\text{ii) } \quad u &= e^{x^2 - y^2} \cos(2xy) \\
\text{iii) } \quad u &= \frac{\sin 2x}{\cosh 2y - \cos 2x}
\end{align*}

satisfies the Laplace equation in two dimensions. In each case, taking \( u \) to be the real part of an analytic function, use the Cauchy-Riemann equations to find \( v(x, y) \), and hence find \( f = u + i v \). (In each case, express \( f \) eventually entirely in terms of \( z \).)

Due Thursday 3rd November in class