(1a) Obtain the recursion relation for the coefficients $a_n$ in the series expansion $y = \sum_{n \geq 0} a_n x^n$ for the equation $y'' - 2xy' + \lambda y = 0$. How many linearly-independent solutions do you obtain by this method?

(1b) Use the ratio test to determine the range of values of $x$ for which the series converges. For what values of $\lambda$ do you obtain series that terminate with a finite number of terms? How many terminating solutions are there for each such $\lambda$?

(2a) Using the definition of the spherical harmonics in terms of $P^m_\ell(\cos \theta)$, and using the expression for $P^m_\ell(x)$ in terms of $P_\ell(x)$ derived in the lectures, show that if

$$L_+ \equiv e^{i\phi} \left( \frac{\partial}{\partial \theta} + i \cot \theta \frac{\partial}{\partial \phi} \right),$$

then

$$L_+ Y_{\ell m}(\theta, \phi) = \sqrt{(\ell - m)(\ell + m + 1)} Y_{\ell,m+1}(\theta, \phi).$$

(It may be more convenient to use the variable $x = \cos \theta$ for this calculation.)

(2b.) Calculate also $L_- Y_{\ell m}(\theta, \phi)$, where $L_-$ is the complex conjugate of $L_+$.

(3a) By using Rodrigues’ formula to define the Legendre polynomials, show that

$$P_\ell(0) = \frac{\ell!(-1)^{\ell/2}}{[\ell/2]!^2} \frac{\ell^\ell}{2^\ell},$$

if $\ell$ is even, while $P_\ell(0) = 0$ if $\ell$ is odd.

(3b) Calculate $P_\ell(0)$ by instead using the generating function.

(3c) Using the generalised Rodrigues formula for the associated Legendre functions, calculate $P^m_\ell(0)$, for all integers $\ell \geq 0$ and all integers $m$ with $m - \ell \leq m \leq \ell$. 

Due Tuesday 20th September in class