(1a) Consider a massive object that is in motion (not necessarily geodesic) inside a Schwarzschild black hole of mass $M$. Using Schwarzschild coordinates $(t, r, \theta, \varphi)$, show that the $r$ coordinate of the object must decrease at a minimum rate given by

$$\left| \frac{dr}{d\tau} \right| \geq \sqrt{\frac{2M}{r} - 1},$$

where $\tau$ is the proper time.

(1b) Hence calculate the maximum possible time that the object can survive, starting at the event horizon, before it hits the singularity at $r = 0$.

(1c) What is the trajectory that achieves this maximum lifetime?

(2) Derive the expressions given in eqn (11.23) in the lecture notes for the Christoffel connection for the metric

$$ds^2 = -N^2 dt^2 + h_{ij} (dx^i + N^i dt)(dx^j + N^j dt).$$

(Extra credit for finding any typos or other mistakes in (11.23)!)