(1a) Recalculate the components of the Christoffel connection $\Gamma^\mu_{\nu \rho}$ for the unit $S^3$ metric

$$ds^2 = d\theta^2 + \sin^2 \theta d\phi^2 + \cos^2 \theta d\psi^2$$

in problem sheet 2, but now using the short cut discussed in section 5.3 of the notes, via the Euler-Lagrange equations for geodesics.

(1b) Recalculate the Christoffel connection components $\Gamma^\mu_{\nu \rho}$ for the $n$-dimensional metric, again from problem sheet 2,

$$ds^2 = dr^2 + e^{2r} \eta_{ab} dy^a dy^b$$

where $\eta_{ab} dy^a dy^b$ is an $(n - 1)$-dimensional Minkowski metric, but using the Euler-Lagrange equations for geodesics.

(2a) Using the Euler-Lagrange method, calculate the Christoffel connection for the metric

$$ds^2 = dr^2 + e^r (dx^2 + dy^2) + e^{2r} (d\psi - x dy)^2.$$ 

Use the coordinate labelling $x^0 = r$, $x^1 = x$, $x^2 = y$, $x^3 = \psi$. (This is actually an Einstein metric on a space that is a non-compact version of the complex projective plane. You are not asked to calculate the curvature here.)

(2b) An antisymmetric tensor $J_{\mu \nu}$ is defined on this space by specifying the components

$$J_{12} = e^r, \quad J_{03} = -e^r, \quad J_{02} = x e^r.$$

Prove that $\partial_{[\mu} J_{\nu \rho]} = 0$. (Note that there are only four separate cases to check here.)

(2c) Optional: Show that $J$ satisfies $J^{\mu \nu} J^\nu_\rho = -\delta^\mu_\rho$.

(2d) Optional: Show that the tensor $J_{\mu \nu}$ in fact obeys a much stronger condition than in part (2b), i.e. that it is covariantly constant, $\nabla_\mu J_{\nu \rho} = 0$. This can be verified by making use of the results for the Christoffel connection obtained in part (2a). $J$ is called the Kähler form. Try doing the calculation, if you want some practice.

(2e) Optional: Calculate the Ricci tensor, and show that the metric is an Einstein metric. (Hint: Directly set $\rho = \mu$ in the expression (4.65) in the notes, to obtain $R_{\mu \nu}$, i.e. do not first calculate all the components of the Riemann tensor! You can also assume the known result that $R_{\mu \nu}$ is symmetric, so it is only necessary to calculate for $\mu \leq \nu$.)

(3) Obtain the equations for geodesics on the unit 2-sphere, using the standard spherical polar coordinates $(\theta, \varphi)$. Prove that a geodesic that starts at $\theta = \frac{1}{2} \pi$ (the equator) with $\dot{\theta} = 0$ remains at $\theta = \frac{1}{2} \pi$, circling the equator. (The dot means $\dot{d}/ds$ here.)

(4) Consider the metric $ds^2 = d\theta^2 + \sin^2 \theta d\varphi^2$ on the unit 2-sphere. Calculate the metric in terms of new coordinates $x = \theta \cos \varphi$ and $y = \theta \sin \varphi$, and show that these are local inertial coordinates in the vicinity of the north pole, where $x$ and $y$ are small. Expand the metric in powers of $x$ and $y$ so as to include the leading order and the first non-trivial order beyond that. Show that the metric at $x = y = 0$ is the Euclidean metric, and that the first derivatives of the metric vanish at $x = y = 0$.

Due in class on Monday 12th October