(1a) Show that the Kronecker delta $\delta^\mu_\nu$ is an invariant tensor under general coordinate transformations.

(1b) Making use of the result (4.60) in the lecture notes for the covariant divergence of a vector field, calculate the Laplacian acting on a scalar field $\psi$, i.e. $\nabla_\mu \partial^\mu \psi$, in the Euclidean 3-metric written in spherical polar coordinates, $ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$.

(2a) Calculate the components of the Christoffel connection $\Gamma^\mu_{\nu\rho}$ for the three-dimensional metric

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta d\psi^2$$

(This metric is the standard metric on a unit-radius 3-sphere.)

(2b) Calculate the components of the Riemann tensor and Ricci tensor, and the Ricci scalar, for the metric in (2a). Show that it is an Einstein metric, $R_{\mu\nu} = \Lambda g_{\mu\nu}$, and determine the value of the constant $\Lambda$.

(3a) Calculate the Christoffel connection components $\Gamma^\mu_{\nu\rho}$ for the $n$-dimensional metric

$$ds^2 = dr^2 + e^{2r} \eta_{ab} dy^a dy^b$$

where $\eta_{ab} dy^a dy^b$ is an $(n-1)$-dimensional Minkowski metric. Use coordinates $x^a = y^a$ for $a = 0, 1, \cdots n - 2$ and $x^{n-1} = r$. (You may find it convenient to give the name $x^r$ to the $x^\mu$ coordinate with $\mu = n - 1$.) (This metric is the metric on $n$-dimensional anti-de Sitter spacetime.)

(3b) Calculate the components of the Riemann tensor, and the Ricci tensor, and also the Ricci scalar, for the metric in (3a). Show that it is an Einstein metric, $R_{\mu\nu} = \Lambda g_{\mu\nu}$, and calculate the constant $\Lambda$.

(4a) Calculate the effect of parallel propagating a vector around a complete circle on the line of latitude $\theta$ = constant, in the metric $ds^2 = d\theta^2 + \sin^2\theta d\phi^2$ on the unit 2-sphere. Take the vector to be in the direction $\partial / \partial \theta$ at $\varphi = 0$, and then use the equation of parallel propagation (eqn (4.102) in the notes) to calculate it after taking it around a full circle, at $\varphi = 2\pi$. Show that it undergoes a rotation, and calculate the rotation angle.

(4b) Consider instead the metric $ds^2 = dr^2 + r^2 d\theta^2$, and calculate the parallel propagation of a vector around a complete circle at $r$ = constant, starting at $\theta = 0$ and ending up at $\theta = 2\pi$. Take the initial vector, at $\theta = 0$, to be in the direction of $\partial / \partial r$. Show that the vector after parallel propagation around the complete circle is exactly the same as the one at the start of the circle.

Due in class on Monday 28th September