(1a) The Riemann tensor in $n$ dimensions has $\frac{1}{12}n^2(n^2 - 1)$ algebraically-independent components. Use this information to show that in two dimensions, the Riemann tensor for any metric can be identically written as

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}R(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}),$$

where $R$ is the Ricci scalar. (Be sure to present an argument for why the Riemann tensor must be of the form (1).)

(1b) Hence show that in two dimensions, the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$$

vanishes identically.

(2a) Consider the four-dimensional metric

$$ds^2 = -e^{2u}dt^2 + e^{-2u}h_{ij}dx^i dx^j,$$

where $u$ and $h_{ij}$ are functions of $x^i$ (with $i = 1, 2, 3$) only. Calculate the Christoffel connection $\Gamma^\mu_{\nu\rho}$ (where $\mu = (0, i)$ and $x^0 = t$), expressing your answers (for $\Gamma^0_{00}$, $\Gamma^0_{0i}$, etc.) in terms of the function $u$ and its derivatives, the 3-metric $h_{ij}$ and the Christoffel connection $\tilde{\Gamma}^{ijk}$ of the 3-metric $h_{ij}$.

(2b) Hence calculate the components of the Ricci tensor $R_{\mu\nu}$ of the four-dimensional metric, expressing your answers in terms of $u$ and its derivatives, $h_{ij}$ and $R_{ij}$ (the Ricci tensor of the 3-metric $h_{ij}$). Your answers should be expressed in terms of 3-dimensionally covariant quantities, so you may need to introduce the covariant derivative $\tilde{\nabla}_i$ constructed using the Christoffel connection $\tilde{\Gamma}^{ijk}$ for the 3-metric $h_{ij}$.

(2c) Hence show that the vacuum Einstein equations imply $\tilde{R}_{ij} = 2\partial_i u \partial_j u$ and $\tilde{\nabla}^2 u = 0$, where $\tilde{\nabla}^2 = \tilde{\nabla}^i \tilde{\nabla}_i$ is the scalar Laplacian calculated in the 3-metric $h_{ij}$.

(3a) A two-dimensional surface $\Sigma$ in Euclidean 3-space $\mathbb{R}^3$ is specified by the embedding

$$x = (a + b \sin \phi) \cos \psi, \quad y = (a + b \sin \phi) \sin \psi, \quad z = b \cos \phi,$$

where $\phi$ and $\psi$ both have period $2\pi$, and $(x, y, z)$ are the Cartesian coordinate in $\mathbb{R}^3$. The constants $a$ and $b$ are such that $a > b > 0$. Sketch the 2-surface $\Sigma$ and describe its shape; i.e. what does it look like?.

(3b) Calculate the induced metric on the 2-surface $\Sigma$.

(3c) Calculate the Ricci tensor for the 2-metric on $\Sigma$.

(3d) Calculate $\int \sqrt{g} R d^2x$ for this metric.

**TURN OVER FOR QUESTION 4 !!!!**
Use the Komar formula

\[ J = \frac{1}{32\pi} \int_{S^2} \epsilon_{\mu \nu} \partial_{\rho} L_{\sigma} d\Sigma^{\mu \nu}, \]  

(2)
evaluated over the "sphere at infinity," to calculate the angular momentum of the Kerr black hole, where \( L = \partial / \partial \varphi \) is the Killing vector that generates azimuthal rotations. (The Kerr metric, and all definitions needed in (2), are given in the notes.)

Note: The answer is already given in the notes; the point of this problem is to give a clearly-presented logical sequence of steps that shows how the result is obtained. Merely reporting that the answer is \( J = am \) is not enough!!!! A very useful tip is to keep in mind is that eventually one is going to take the limit where \( r \) goes to infinity, and so one can expand the Kerr metric in inverse powers of \( r \) at the outset and keep only the leading-order term in each component.

Please scan your script, and e-mail it to me as a single pdf file, by the deadline on Thursday at 5:00pm. Please make sure you write using a pen that will be legible in the scanned version!