611: Electromagnetic Theory

Problem Sheet 6

(1a) It was shown in the lectures that the potentials describing an electric charge \( e \) moving with constant velocity \( \vec{v} \) are given by

\[
\phi(\vec{r}, t) = \frac{e \gamma}{r'}, \quad \vec{A}(\vec{r}, t) = \vec{v} \phi, \tag{1}
\]

where \( r' \) is to be re-expressed in terms of the unprimed coordinates using the usual Lorentz boost formula. Using just the relation \( \vec{A} = \vec{v} \phi \), show that the magnetic field can be written as \( \vec{B} = \vec{v} \times \vec{E} \).

(1b) Now using the detailed expressions for \( \phi \) and \( \vec{A} \) in (1), calculate the electric field \( \vec{E} = -\vec{\nabla} \phi - \partial \vec{A}/\partial t \), and show that it agrees with the expression derived in the lectures.

(2) Consider the expression for the electric field due to a charge \( e \) moving with uniform velocity \( \vec{v} \), as derived in the lectures. Evaluate the surface integral

\[
\int_S \vec{E} \cdot d\vec{S}
\]

over a spherical surface that encloses the moving charge.

(3) The angular momentum 3-vector \( \vec{L} \) is defined by \( L_i = \frac{1}{2} \epsilon_{ijk} M_{jk} \), where \( M^{\mu \nu} = \int (x^\mu T^{\nu \rho} - x^\nu T^{\mu \rho}) d\Sigma_\rho \) is the angular momentum 4-tensor as defined in the lectures.

(3a) Prove from the above that for the electromagnetic field,

\[
\vec{L} = \frac{1}{4\pi} \int \vec{r} \times (\vec{E} \times \vec{B}) d^3 x
\]

(3b) Prove that if the fields die off appropriately at infinity, this can be written (assuming there are no sources, so \( J^\mu = 0 \)) as

\[
\vec{L} = \frac{1}{4\pi} \int [(\vec{E} \times \vec{A}) + E_i (\vec{r} \times \nabla) A_i] d^3 x
\]

(Note that the second term is like an “orbital” contribution, whose value depends on the choice of origin of the coordinate system, whereas the first term is like an “intrinsic spin” contribution that does not depend on the choice of origin.)

Due on Monday 3rd November