(1a) Show that if a tensor $T_{\mu\nu}$ is antisymmetric in one Lorentz frame, then it is antisymmetric in all Lorentz frames.

(1b) Similarly, show that if $T_{\mu\nu}$ is symmetric in one frame, it is symmetric in all frames.

(1c) Show that if a tensor $A_{\mu\nu}$ is antisymmetric, then $A^{\mu\nu}$ is also antisymmetric.

(1d) Show that the tensor $S_{\mu\nu} = F_{\mu\rho} F_{\nu}^{\rho}$ is always symmetric, regardless of any symmetry properties of $F_{\mu\nu}$.

(2) A 4-vector $V^\mu$ is called timelike if $V^\mu V_\mu < 0$; spacelike if $V^\mu V_\mu > 0$; and null (or lightlike) if $V^\mu V_\mu = 0$. Suppose that $k^\mu$ is a null vector. Show that if a non-spacelike vector $V^\mu$ is orthogonal to $k^\mu$ (i.e. $k^\mu V_\mu = 0$), then it must be that $V^\mu$ is just a multiple of $k^\mu$.

(3a) Consider an infinitesimal Lorentz transformation for which $\Lambda^\mu_\nu = \delta^\mu_\nu + \lambda^\mu_\nu$, where $\lambda^\mu_\nu$ is infinitesimally small. Write down the expression for the infinitesimal change in the spacetime coordinates, $\delta x^\mu = x'^\mu - x^\mu$, in terms of $\lambda^\mu_\nu$.

(3b) Show from the defining equation for Lorentz transformations, $\eta_{\mu\nu} \Lambda^\mu_\rho \Lambda^\nu_\sigma = \eta_{\rho\sigma}$, that $\lambda^\mu_\nu$ satisfies $\lambda_{\mu\nu} = -\lambda_{\nu\mu}$.

(3c) Define the quantities $M_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu$. Show that acting on $x^\mu$, we have

$$\frac{1}{2} \lambda^\mu_\sigma M_{\rho\sigma}(x^\mu) = -\lambda^\mu_\nu \partial_\mu x'^\nu = -\delta x'^\mu.$$

(3d) The calculation in (3c) shows that the $M_{\mu\nu}$ are the generators of the Lorentz group $O(1, 3)$. Show that they satisfy the algebra

$$[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\mu\rho} M_{\nu\sigma} - \eta_{\mu\sigma} M_{\nu\rho} + \eta_{\nu\sigma} M_{\mu\rho} - \eta_{\nu\rho} M_{\mu\sigma}.$$