611: Electromagnetic Theory

Problem Sheet 4

(1a) The angular momentum 3-vector $\vec{L}$ is defined by $L_i = \frac{1}{2} \varepsilon_{ijk} M_{jk}$, where $M^{\mu\nu} = \int (x^\mu T^{\nu\rho} - x^\nu T^{\mu\rho}) d\Sigma_\rho$ is the angular momentum 4-tensor as defined in the lectures.

(a) Prove from the above that for the electromagnetic field,

$$\vec{L} = \frac{1}{4\pi} \int \vec{r} \times (\vec{E} \times \vec{B}) d^3 x$$

(b) Prove that if the fields die off appropriately at infinity, this can be written (there are no sources, so $J^\mu = 0$) as

$$\vec{L} = \frac{1}{4\pi} \int [\vec{E} \times \vec{A} + E_i (\vec{r} \times \vec{\nabla}) A_i] d^3 x$$

(2) Show that the conservation law for the $M^{0i}$ components of $M^{\mu\nu}$ implies that

$$\frac{d\vec{R}}{dt} = \frac{\vec{P}}{\mathcal{E}}$$

where $\vec{R}$ is the centre of mass of the electromagnetic field, defined by $\vec{R} \int W d^3 x = \int \vec{r} W d^3 x$. Here $W$ is the energy density, $\mathcal{E}$ is the total energy and $\vec{P}$ is the total 3-momentum.

(3) Derive the equations of motion (Euler-Lagrange equations) that follow from the Lagrangian density

$$\mathcal{L} = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} - \frac{m^2}{8\pi} A^\mu A_\mu + J^\mu A_\mu$$

where $m$ is a constant, and $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$. This theory is a massive generalisation of Maxwell’s theory, known as the Proca theory. Find the solution for the scalar potential $\phi \equiv A_0$ describing a “point charge” $q$ located at the origin. (i.e. find the analogue in Proca theory of the usual static, spherically symmetric point charge in ordinary electrodynamics.)

Due on Thursday 14’tth October