(1a) Consider two successive Lorentz boosts along the x axis, one with velocity \( v_1 \) and the other with velocity \( v_2 \). Show that the two boosts commute; i.e. the final result is the same whether one makes the \( v_1 \) boost first followed by the \( v_2 \) boost, or instead one makes the \( v_2 \) boost first followed by the \( v_1 \) boost.

(1b) Show that the result of making the two successive boosts is again a Lorentz boost along the x axis, and obtain the expression for the net boost velocity \( v_3 \) for this transformation.

(1c) Show that for successive boosts with \( \vec{v}_1 = (v_1, 0, 0) \) along the x axis and \( \vec{v}_2 = (0, v_2, 0) \) along the y axis, the transformations do not commute.

(1d) Explain why the resulting transformations in part (c) can be seen to be not pure boosts, but that they involve spatial rotations also.

(2) Complete the derivation of the discussion in section 2.4 of the lecture notes, by deriving the Lorentz transformation that gives \( \vec{B}' \) in terms of \( \vec{E} \) and \( \vec{B} \), for an arbitrary Lorentz boost with velocity \( \vec{v} \).

(3a) Consider an infinitesimal Lorentz transformation for which \( \Lambda^\mu_\nu = \delta^\mu_\nu + \lambda^\mu_\nu \), where \( \lambda^\mu_\nu \) is infinitesimally small. Write down the expression for the infinitesimal change in the spacetime coordinates, \( \delta x^\mu = x'^\mu - x^\mu \), in terms of \( \lambda^\mu_\nu \).

(3b) Show from the defining equation for Lorentz transformations, \( \eta_{\mu\nu} \Lambda^\mu_\rho \Lambda^\nu_\sigma = \eta_{\rho\sigma} \), that \( \lambda^\mu_\nu \) satisfies \( \lambda^{\mu\nu} = -\lambda^{\nu\mu} \).

(3c) Define the vector fields \( M_{\mu\nu} = x^\mu \partial_\nu - x^\nu \partial_\mu \). Show that acting on \( x^\mu \), we have

\[
\frac{1}{2} \lambda_{\rho\sigma} M_{\rho\sigma}(x^\mu) = \lambda^{\mu}_\nu x^\nu = \delta x^\mu.
\]

(3d) The calculation in (3c) shows that the \( M_{\mu\nu} \) are the generators of the Lorentz group \( O(1,3) \). Show that they satisfy the algebra

\[
[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\mu\rho} M_{\nu\sigma} - \eta_{\nu\rho} M_{\mu\sigma} + \eta_{\nu\sigma} M_{\mu\rho} - \eta_{\mu\sigma} M_{\nu\rho}.
\]

As always, you should be careful not to “over use” dummy indices when solving problems 2 and 3!

Due on Tuesday 21st September