**603: Electromagnetic Theory I**

Problem Sheet 4

(1a) A thin, flat disc of radius $a$ is located in the $x-y$ plane with its centre at the origin. It has a surface charge density $\sigma = k(a^2 - \rho^2)^{-1/2}$, where $\rho$ is the distance out from the centre of the disc and $k$ is a constant. Calculate (by elementary methods) the potential on the $z$ axis, both for positive $z$ and negative $z$.

(1b) Show, by using the method of off-axis extrapolation, that the potential at points $(r, \theta, \varphi)$ with $r > a$ is given by

$$\phi(r, \theta, \varphi) = \frac{2\pi ka}{r} \sum_{n \geq 0} \frac{(-1)^n}{2n + 1} \left(\frac{a}{r}\right)^{2n} P_{2n}(\cos \theta)$$

(1c) Find the analogous series expansion that is valid for $r < a$, again by using the method of off-axis extrapolation. Do this both for the region above the plane of the disc and the region below the plane of the disc.

(2a) Use your result in part (1c) to calculate the potential at all points on the surface of the disc. (Recall that $P_{\ell}(x) = (-1)^\ell P_{\ell}(-x)$, so $P_{\ell}(0) = 0$ if $\ell$ is odd.)

(2b) Calculate the total charge on the disc. Hence, making use of your result from part (2a), calculate the capacitance of a thin conducting disc of radius $a$.

(2c) Compare the expansions in (1b) and (1c), and discuss this in the context of the “solution by inversion” procedure described in section 4.6 of the lectures.

(3a) Use the recursion relation for the $a_n$ coefficients in the Frobenius expansion of the solutions of the Legendre equation to find the $a_n$ for the even solution with $\ell = 1$ (i.e. $\lambda = \ell(\ell + 1) = 2$).

(3b) Show that this even solution can be written in closed form as

$$y(x) = a_0 \left[1 + \frac{1}{2} x \log \left(\frac{1 - x}{1 + x}\right)\right],$$

and hence that it diverges at $x = \pm 1$.

Due on Wednesday 24th February, in class