603: Electromagnetic Theory I
Problem Sheet 10

(1a) Consider a solenoid of radius $a$ wound with $N$ turns per unit length, carrying a current $I$. Suppose the $z$-axis runs along the axis of the cylinder, and that it is of length $L$, with $-\frac{1}{2}L \leq z \leq \frac{1}{2}L$. Show that when $N$ is large, the magnetic field on the axis is in the $z$ direction, with

$$B_z = \frac{2\pi NI}{c} (\cos \theta_1 + \cos \theta_2),$$

where $\theta_1$ and $\theta_2$ are indicated in the diagram below:

(1b) For a long solenoid where $L \gg a$, show that near the axis and near the midpoint of the solenoid (i.e. $\rho \ll a$ and $|z| \ll L/2$, in cylindrical polar coordinates), the magnetic field is mainly parallel to the $z$ axis, but has a small radial component

$$B_\rho \approx \frac{96\pi NI \ a^2 z \rho}{L^4}.$$

(Hint: Use $\nabla \cdot \vec{B} = 0$.)

(1c) Show that near the end of a long solenoid, the magnetic field near the axis has

$$B_z \approx \frac{2\pi NI}{c}, \quad B_\rho \approx \pm \frac{\pi NI \ \rho}{a}.$$

(2) A circular current loop of radius $a$ carrying a current $I$ lies in the $x-y$ plane, centred on the origin. Show that the only non-vanishing component of the vector potential, in cylindrical polar coordinates, is

$$A_\varphi (\rho, z) = \frac{4Ia}{c} \int_0^\infty dk \ \cos kz \ I_1(k\rho_<) \ K_1(k\rho_>)$$

(3a) Use the suffix notation for three-dimensional Cartesian vectors, and the $\epsilon_{ijk}$ tensor, to prove the identities

1. $\nabla \times \nabla \phi \equiv 0$ for any function $\phi(\vec{r})$,
2. $\nabla \cdot (\nabla \times A) \equiv 0$ for any vector $A(\vec{r})$
3. $\nabla \times (\nabla \times A) = \nabla (\nabla \cdot A) - \nabla^2 A$ for any vector $A(\vec{r})$.

(3b) Show that for any vector $\vec{V}$,

$$\partial_i V_j - \partial_j V_i = \epsilon_{ijk} W_k,$$

where $\vec{W} = \nabla \times \vec{V}$.

Due on Wednesday 27th April