603: Electromagnetic Theory I

Problem Sheet 1

(1) Show by explicit direct substitution, making use of the result that \( \nabla^2|\vec{r} - \vec{r}'|^\text{-}1 = -4\pi \delta(\vec{r} - \vec{r}') \), that if

\[
\phi(\vec{r}) = \int \frac{\rho(\vec{r}') d^3\vec{r}'}{|\vec{r} - \vec{r}'|},
\]

then \( \phi \) satisfies \( \nabla^2 \phi(\vec{r}) = -4\pi \rho(\vec{r}) \).

(2a) A spherically-symmetric charge distribution gives rise to the potential

\[
\phi = \frac{q e^{-\mu r}}{r} \left( 1 + \frac{\mu r}{2} \right)
\]

everywhere, where \( q \) and \( \mu \) are constants. Find the charge distribution \( \rho(r) \) that gives rise to this potential. (Be sure to consider \( r = 0 \) carefully, as well as \( r > 0 \). i.e., consider delta-function terms also.)

(2b) Integrate your result for \( \rho(\vec{r}) \) (including both the bulk and delta-function terms) over the volume of all space, and hence obtain the total electric charge. Discuss why this result is reasonable. (Consider a Gauss’ law integral of \( \int \vec{E} \cdot d\vec{S} \) at infinity.)

(3a) Substitute the expressions \( \vec{E} = -\vec{\nabla} \phi - (1/c)\partial \vec{A} / \partial t \) and \( \vec{B} = \vec{\nabla} \times \vec{A} \) into the Maxwell field equations in vacuum,

\[
\vec{\nabla} \cdot \vec{E} = 4\pi \rho, \quad \vec{\nabla} \times \vec{B} - \frac{1}{c} \frac{\partial \vec{E}}{\partial t} = \frac{4\pi}{c} \vec{J},
\]

and hence obtain the equations satisfied by the potentials \( \phi \) and \( \vec{A} \).

(3b) Suppose now that the gauge potentials are in Lorenz Gauge, which means they obey \( \vec{\nabla} \cdot \vec{A} + (1/c)\partial \phi / \partial t = 0 \). What do the equations you obtained in part (3a) now become?

(3c) Suppose we begin with gauge potentials \( \phi \) and \( \vec{A} \) that are not in Lorenz gauge. Derive a differential equation for the gauge transformation parameter \( \lambda \) such that \( \phi' \) and \( \vec{A}' \), defined by

\[
\phi' = \phi - \frac{1}{c} \frac{\partial \lambda}{\partial t}, \quad \vec{A}' = \vec{A} + \vec{\nabla} \lambda,
\]

are in Lorenz gauge.

Due on Wednesday 4th February