603: Electromagnetic Theory I

Problem Sheet 6

(1) Many properties of the Bessel functions can be derived from a generating function. This defines the $J_n(x)$ (for $n$ an integer) as follows:

$$G(x, t) \equiv e^{\frac{1}{2}x^2(t-1)} = \sum_{n=-\infty}^{\infty} t^n J_n(x)$$

All the following problems should be solved using this definition.

(1a) Show that $J_n(x)$ as defined above does indeed satisfy Bessel's equation $x^2 J''_n(x) + x J'_n(x) + (x^2 - n^2) J_n(x) = 0$. (The proof is analogous to the one given in the lectures for the Legendre polynomials.)

(1b) Show that $J_{n-1}(x) - J_{n+1}(x) = 2J'_n(x)$.

(1c) Show that $J_{n-1}(x) + J_{n+1}(x) = (2n/x)J_n(x)$.

(1d) Show that $J_{-n}(x) = (-1)^n J_n(x)$.

(2) A hollow right cylinder of radius $a$ has its axis along the $z$ axis, and it has end caps at $z = 0$ and $z = h$. The potential on the end caps is zero, and on the cylindrical surface at $\rho = a$ it is given by $\phi(a, \varphi, z) = V(\varphi, z)$, where $V(\varphi, z)$ is a given function. By making the appropriate separation of variables in cylindrical coordinates, find a series solution for the potential everywhere inside the cylinder. (Obtain expressions, as integrals involving $V(\varphi, z)$, for the coefficients in the series solution.)

(3a) By taking $t \to i e^{i \varphi}$, $x \to k \rho$, use the generating function for Bessel functions in question (1) to prove that

$$e^{ik \rho \cos \varphi} = \sum_{n=-\infty}^{\infty} i^n e^{n \varphi} J_n(k \rho).$$

(3b) Show (using an elementary calculation in Cartesian coordinates) that $\phi = e^{k(z+ix)}$ satisfies Laplace's equation (now, in this part of the problem, $x$ and $z$ are Cartesian coordinates). Writing $\phi$ in cylindrical polar, show using the result in part (3a) that $\phi$ is a special case of the general form of the solutions discussed in section 5 of the lectures.

Due on Wednesday 19th March