603: Electromagnetic Theory I
Problem Sheet 5

(1) The potential on the surface \( r = a \) is given to be

\[
\phi(a, \theta, \varphi) = V_0 \sin \theta \cos \theta \sin \varphi. \tag{1}
\]

Obtain two fully explicit expressions for \( \phi(r, \theta, \varphi) \), valid in the two regions \( r > a \) and \( r < a \) respectively. (Hint: Using the expressions for the first few spherical harmonics given in the lectures, first express the boundary value (1) in terms of the \( Y_{\ell m}(\theta, \varphi) \).

(2a) In Cartesian coordinates, show that if \( \vec{p} \) is a constant vector, then \( \phi \equiv \vec{p} \cdot \nabla (1/r) \) satisfies Laplace’s equation for \( r > 0 \).

(2b) Obtain the expression for the potential \( \phi \) in part (2a) as a fully explicit expansion in spherical harmonics. [Note: The answer involves just a finite number (small!) of terms. The coefficients are completely explicit.]

(3a) Consider the angular momentum generators

\[
L_1 = -i(y\partial_z - z\partial_y), \quad L_2 = -i(z\partial_x - x\partial_z), \quad L_3 = -i(x\partial_y - y\partial_x),
\]

(where \( \partial_x \equiv \partial/\partial x \), etc.) Using the relations \( x = r \sin \theta \cos \varphi \), \( y = r \sin \theta \sin \varphi \), \( z = r \cos \theta \), show that

\[
L_+ = e^{i\varphi} \left( \frac{\partial}{\partial \theta} + i \cot \theta \frac{\partial}{\partial \varphi} \right), \quad L_3 = -i \frac{\partial}{\partial \varphi},
\]

where \( L_\pm \equiv L_1 \pm i L_2 \).

(3b) Using the generalised Rodrigues formula for the associated Legendre functions \( P^m_\ell \), and the definition of the spherical harmonics, derive expressions for \( L_3 Y_{\ell m} \), \( L_+ Y_{\ell m} \) and \( L_- Y_{\ell m} \). (In each case, the answer is a constant factor times a single spherical harmonic.)

Due on Wednesday 26th February