(1) Calculate the monopole $Q$, dipole $p_i$ and quadrupole $Q_{ij}$ moments, defined in eqn (6.30) of the lectures, for each of the following two charge distributions:

(1a) Charge $q$ at $\vec{r} = (a, 0, 0)$; charge $-q$ at $\vec{r} = (-a, 0, 0)$; charge $q$ at $\vec{r} = (0, a, 0)$; and charge $-q$ at $\vec{r} = (0, -a, 0)$. (i.e. four point charges in total.)

(1b) Charge $q$ at $\vec{r} = (0, 0, a)$; charge $q$ at $\vec{r} = (0, 0, -a)$; and charge $-2q$ at $\vec{r} = (0, 0, 0)$. (i.e. three point charges in total.)

(2a) For any charge distribution $\rho(\vec{r})$, prove that the values of the $(2\ell + 1)$ independent components of the first non-zero multipole moment are independent of the choice of origin.

(2b) A charge distribution has monopole, dipole and quadrupole moments $Q$, $p_i$ and $Q_{ij}$ for some given choice of origin for Cartesian coordinates $x_i$. With respect to another system $x'_i$ of Cartesian coordinates, parallel to the $x_i$ but with origin located at $x_i = a_i$ ($a_i$ are given constants), the corresponding multipole moments are $Q'$, $p'_i$ and $Q'_{ij}$. Obtain explicit expressions for $Q'$, $p'_i$ and $Q'_{ij}$ in terms of $Q$, $p_i$, $Q_{ij}$ and the displacement vector $a_i$.

(3) A charge distribution is given in spherical polar coordinates by

$$\rho = \frac{1}{64\pi} r^2 e^{-r} \sin^2 \theta.$$  

Calculate all its multipole moments $q_{\ell m}$ (as defined by eqn (6.44) in the lectures).

Due on Tuesday 16th April